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**Abstract**

In a single-commodity, pure-exchange, representative-agent economy with many Lucas' trees whose dividends are geometric Brownian motions, I study the comparative statics of the prices of these assets with respect to the current Brownian realization. As is well-known, due to wealth effects, a security's price may vary with the realization of a Brownian motion even when its dividend is independent of it. Yet, a crucial component of wealth effects has hitherto been ignored by the literature: changes in wealth do not alter only the agent's risk aversion, but also her perceived "riskiness" of the security. This enhances the extent to which market-clearing leads to endogenously-generated correlation across asset prices and returns, over and above that induced by correlation between payoffs, giving the appearance of "contagion". I establish also a necessary and sufficient condition for the securities market to be dynamically complete. Being independent of the utility function of the representative agent, it applies even in the presence of many heterogenous agents.

**Keywords**: General Equilibrium, Comparative Statics, Contagion, Dynamically Complete Markets.

**JEL Classification Numbers**: G10, G12.

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# 1 Introduction

For the most part of the theoretical continuous-time financial economics literature, the workhorse has been some analogue of the model in Lucas [27]. In its purest form, it depicts a one-commodity, pure-exchange economy with identical price-taking consumers, in which economic activity occurs over a time-interval $[0, T] \subseteq \mathbb{R}_+$. The good is produced by $N \in \mathbb{N}^*$ distinct units whose productivity fluctuates stochastically, their usual interpretation being that of Lucas trees. Namely, a crop is growing stochastically on different trees via a production process that is entirely exogenous: no resources are utilized and there is no possibility of affecting the output of any tree at any time.

The magnitude of the crop plays the role of an information process. It is monitored by all individuals who are continuously revising their beliefs about its future payoff. The typical informational structure takes as primitive a complete probability space $(\Omega, \mathcal{F}, \pi)$ on which is defined a $K$-dimensional (with $K \geq N$) standard Brownian motion $\beta = \{\beta(\omega, t) : t \in [0, T]\}_{\omega \in \Omega}$ and the filtration $\{\mathcal{F}_t : t \in [0, T]\}$ it generates.[1] This is meant to describe the exogenous uncertainty about productivity in the sense that the sample paths in the collection $\{\beta(\omega, [0, T]) :\}_{\omega \in \Omega}$ completely specify all the distinguishable events.

Even though commonly endowed with the generated filtration, the individuals cannot observe $\beta$ directly. Instead of the actual productivity shocks, they monitor the crop on the trees, depicted by the $N$-dimensional process $Y$, which is a function of the process $\mathcal{I} = \{\beta(\omega, t), t\}_{(\omega, t) \in \Omega \times [0, T]}$ and whose

[1] A probability space $(\Omega, \mathcal{F}, \pi)$ consists of a sample space $\Omega$, a $\sigma$-algebra $\mathcal{F}$ on $\Omega$, and a probability measure $\pi$ on $\mathcal{F}$. Each $\omega \in \Omega$ represents a complete description of the exogenous uncertain environment while $\mathcal{F}$ is the collection of the distinguishable, at the end of time, events. The probability space is complete if any subset of any $\pi$-null set is included in $\mathcal{F}$. A filtration $\{\mathcal{F}_t : t \in [0, T]\}$ is a family of $\sigma$-algebras $\mathcal{F}_t \subseteq \mathcal{F}$ which is increasing: $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$. It depicts the evolution of information: $\mathcal{F}_t$ represents the information available at $t$. The filtration being increasing, more and more is known with time (past information is not forgotten). Being, in particular, generated by the Brownian motion, it depicts the informational structure revealed to someone who observes the path of the Brownian motion. Mathematically, this entails $\mathcal{F}_t = \{\beta(\omega, s) : (\omega, s) \in \Omega \times [0, t]\}$ while $\mathcal{F}_T = \mathcal{F}$. We assume that $\beta(\omega, 0) = \mathbf{0}\ \forall \omega \in \Omega$ almost surely, so that $\mathcal{F}_0$ is almost trivial (it contains only $\Omega$ and all the $\pi$-null sets).

component processes $Y_1, \ldots, Y_N$ represent the current amount of the consumption good on the respective tree. Of course, the evolution of $Y$ over time depends upon $\beta$ in a nonpredictable fashion, being adapted to the given filtration.[2]

The trading structure in the basic model consists of $N+1$ perfectly divisible securities, which are continuously and frictionlessly traded in a market. Each security $n \in \{1, \ldots, N\}$ represents one equity share (termed "stock") in the $n$th productive unit and is in positive net supply. The remaining security is a promissory note (termed a "bond"), paying one unit of the good with certainty, and in zero net supply. Finally, individual preferences are such that the representative agent has some von-Neumann Morgenstern utility function over consumption, $u : \mathbb{R}_{++} \mapsto \mathbb{R}$, which is twice continuously differentiable, strictly increasing, and concave everywhere in its domain.

In the equilibrium of this economy, at any node $(\omega, t)$, the price $P_n(\omega, t)$ of the typical security $n \in \{0, 1, \ldots, N\}$ is the current expectation of its future dividends valued at the representative agent's marginal rate of substitution between consumption at the dividend-collection date and the present. Derivations have been provided by a number of seminal papers and for different versions of the model (see the next section for more details), which have been then used extensively in the literature to price other financial assets, such as derivative securities, and to identify the optimal consumption and portfolio policies. Surprisingly, though, the dynamics of the equilibrium price processes with respect to the underlying stochastic process have not been thus far investigated; not analytically and, hence, not to a satisfactory degree of generality with respect to the economic primitives. And this is the task of the present paper.

Of course, marginal utilities are not observable in practice and securities

[2] The process $Y$ is said to be adapted to the filtration $\{\mathcal{F}_t : t \in [0, T]\}$ if, for each $\omega \in \Omega$, $Y(\omega, t)$ is is measurable with respect to $\mathcal{F}_t$ $\forall t \in [0, T]$. In words, whatever the underlying true state of the world $\omega$, the value of $Y$ at any date cannot depend on any realization of the Brownian motion after that date. The process $\mathcal{I}$ depicts the vector Brownian process $\beta$ but also time as distinct entities. As functional argument, it allows for time- as well as state-dependence in the corresponding function, as long as the latter dependence obtains only through the realizations of the Brownian process.

are priced with respect to a numeraire, such as dollars. The underlying informational structure being a filtration, however, the choice of numeraire is essentially arbitrary because the equilibrium market-clearing condition depends only on the relative prices of the securities and consumption, and does so node $(\omega, t)$ by node $(\omega, s)$, for $s \neq t$.[3] We may choose, therefore, consumption as the numeraire and set its price at $P_c(\omega, t) = 1\ \forall (\omega, t) \in \Omega \times [0, T]$. My aim then is to investigate the dynamics of $p_n(\omega, t) = \frac{P_n(\omega,t)}{P_0(\omega,t)}$ for $n \in \{1, \ldots, N\}$, the equilibrium relative price process of the typical stock (relative to the price of the bond) with respect to the current realization $\beta_k(\omega, t)$ of the typical Brownian motion.

Whether these dynamics are monotone is the most fundamental comparative statics question. For if (and only if) they are, there can be an invertible relation between the equilibrium relative prices of the assets in this economy and the underlying stochastic process that represents its primitive sources of risk. This would, for instance, greatly facilitate economic but also econometric analysis and prediction. In fact, it would ensure that either (and especially the latter) makes sense by rendering the effect of the unobserved Brownian process on the equilibrium asset prices identifiable from the observable path of the production process, the available information process in this economy.[4]

[3]Recall that each $\omega \in \Omega$ is a complete description of the uncertain environment. As such, it gets predetermined exogenously and remains fixed throughout time. What changes with time is the path of realizations for the underlying stochastic process that generates the filtration $\{\mathcal{F}_t : t \in [0, T]\}$. Being a $K$-dimensional standard Brownian motion, its component processes $\beta_1, \ldots, \beta_K$ are independent, one-dimensional Brownian motions with zero drift and unit variance so that the process changes here in increments such that, for all $0 \leq s < t \leq T$, $\beta(\omega, t) - \beta(\omega, s)$ is independent of $\mathcal{F}_s(\omega)$ and distributed $\mathcal{N}(\mathbf{0}, (t-s)\,\mathrm{I}_K)$. A given $\omega$ determines, therefore, the Brownian path $\beta(\omega, [0, T])$. And since this path has been drawn by nature before the economic activity even starts, the equilibrium market-clearing conditions need to apply only along the path; along every possible path, of course, but not across paths. As a consequence, and given that only relative prices matter in equilibrium, it is without loss of generality to normalize such that the price of one of the traded entities is 1 throughout every path.

[4]Let $\mathrm{d}Y = \mathbf{a}\mathrm{d}t + B\mathrm{d}\beta$ be an $N$-dimensional Ito process and $D \subseteq \mathbb{R}^N$ an open set such that $Y(\omega, t) \in D\ \forall (\omega, t) \in \Omega \times [0, T]$ almost surely. Even though not displayed as such to save on notation, the quantities $\mathbf{a} \in \mathbb{R}^N$ and $B \in \mathbb{R}^{N \times K}$ can be also stochastic as long as $\mathbf{a}(Y(\omega, t), t) \in \mathcal{L}^1$ and $B(Y(\omega, t), t) \in \mathcal{L}^2$. Consider now a twice-differentiable function $f : D \mapsto \mathbb{R}$ (such as any price in the model). By Ito's lemma, and not displaying the

To examine these comparative statics analytically, I use the closed form solution for $p_n(\omega, t)$. This has been provided by two related strands of the literature. One assumes that the crop on the trees is ripe for consumption only at a finite terminal date $T$. At any intermediate time $t \in [0, T)$, the agent consumes some exogenously-given deterministic endowment flow (see, for example, Raimondo [34] as well as Anderson and Raimondo [6]) or nothing at all (as in Bick [8]-[9] but also He and Leland [21]). Letting $W$ denote the representative agent's wealth process (in units of consumption), we have then

$$p_n(\omega, t) = \frac{P_n(\omega, t)}{P_0(\omega, t)} = \frac{\mathbb{E}_\pi\left[u'\left(W\left(\mathcal{I}(\omega, T)\right)\right) D_n\left(\mathcal{I}(\omega, T)\right) | \mathcal{F}_t\right]}{\mathbb{E}_\pi\left[u'\left(W\left(\mathcal{I}(\omega, T)\right)\right) | \mathcal{F}_t\right]} \tag{1}$$

The other approach (see, for instance, Cox et al. [13], Merton [29]-[29], Cochrane et al. [12] or Martin [28]) has been to consider the actual continuous-time extension of the setting in Lucas [27] and grant the agent continuous access to the crop so that her consumption can be financed by the trees' payoffs at all times while $T$ may be infinite. The equilibrium relative price of the $n$th risky security is then essentially the continuous-time analogue of the previous pricing formula[5]

$$p_n(\omega, t) = \frac{\mathbb{E}_\pi\left[\int_t^T u'\left(W\left(\mathcal{I}(\omega, s)\right), s\right) D_n\left(\mathcal{I}(\omega, s)\right) \mathrm{d}s | \mathcal{F}_t\right]}{\mathbb{E}_\pi\left[\int_t^T u'\left(W\left(\mathcal{I}(\omega, s)\right), s\right) \mathrm{d}s | \mathcal{F}_t\right]} \tag{2}$$

To enable the analytical manipulation of these functionals, I will restrict

dependence upon $(\omega, t)$, $\mathrm{d}f(Y) = \left[f_Y(Y)\,\mathbf{a} + \frac{1}{2}\mathrm{tr}\left(B^\intercal f_{YY}(Y)\,B\right)\right]\mathrm{d}t + f_Y(Y)\,B\mathrm{d}\beta$ where $f_Y = \left(\frac{\partial f}{\partial Y_1}, \ldots, \frac{\partial f}{\partial Y_N}\right)$ and $f_{YY} = \left(\frac{\partial^2 f}{\partial Y_i \partial Y_j}\right)_{i,j=1}^N$ denote the gradient vector (in row form) and the Hessian matrix of $f$, respectively. If one fixes time, the "sensitivity" of $f$ with respect to changes in the realization of the underlying Brownian risk factors is given by $\mathrm{d}f(Y) = \sum_{n=1}^N \sum_{k=1}^K \frac{\partial f(Y)}{\partial Y_n} b_{nk}\mathrm{d}\beta_k$. In particular, restricting attention to changes in the $k$th risk source only, $\frac{\partial f(Y)}{\partial \beta_k} = \mathbf{b}_k^\intercal f_Y(Y)$ where $\mathbf{b}_k$ is the $k$th column of $B$.

[5] In fact, regarding the economic underpinnings, the main difference between the two approaches concerns the instantaneous risk-free rate during the intermediate period. The representative agent's endowment and, thus, consumption being deterministic in the intermediate period, the instantaneous risk-free rate is exogenously-specified in the first approach. By contrast, it is derived as part of the equilibrium in the other.

attention to the case in which the typical component of the production process follows a geometric Brownian motion: $Y_n(\mathcal{I}(\omega,t)) = e^{\mu_n t + \sigma_n^\intercal \beta(\omega,t)}$, both the drift $\mu_n \in \mathbb{R}$ and the instantaneous covariance matrix $\sigma_n \sigma_n^\intercal \in \mathbb{R}^{K\times K}$ being constants. This specification is chosen mainly for three reasons. It has been widely used in theoretical as well as empirical studies because it allows the equilibrium asset prices to be identified also as solutions to well-known stochastic differential equations. More importantly for the current study, it provides a setting in which the derivative $\frac{\partial p_n(\omega,t)}{\partial \beta_k(\omega,t)}$ can be recovered from the current information on future dividends in a very straightforward way.[6] And last but by no means least, it greatly facilitates the exposition as it allows us to restrict attention on obtaining insights and results about the dynamics of the pricing process in (1) which are also valid for the dynamics of that in (2).[7]

Even though $p_n(\omega,t)$ can be obtained in closed form, determining its basic comparative statics properties is not straightforward for two reasons. By the quotient rule, the derivative $\frac{\partial p_n(\omega,t)}{\partial \beta_k(\omega,t)}$ is the sum of two terms which may well be of opposite sign. And even the absolute price $P_n(\omega,t)$ may exhibit complex dynamics. Other things being equal, an increase in the $n$th

[6] By Ito's lemma, the current output of the $n$th productive unit follows the Ito process $\mathrm{d}\ln Y_n = \left(\mu_n - \frac{\sigma_n^\intercal \sigma_n}{2}\right)\mathrm{d}t + \sigma_n^\intercal \mathrm{d}\beta$. Hence, for the $N$-dimensional process $X = (\ln Y_n)_{n=1}^N$, we have $\mathrm{d}X = \left(\mu_n - \frac{\sigma_n^\intercal \sigma_n}{2}\right)_{n=1}^N \mathrm{d}t + \Sigma \mathrm{d}\beta$ where $\Sigma$ is the $N \times K$ matrix with $\sigma_n^\intercal$ its typical row. Recall now the one before the preceding footnote. The "sensitivity" of $p_n$ with respect to changes in the realization of the underlying Brownian risk factors is given by $\mathrm{d}p_n(X) = \sum_{n=1}^N \sum_{k=1}^K \frac{\partial p_n(X)}{\partial X_n} \sigma_{nk} \mathrm{d}\beta_k$ so that $\frac{\partial p_n(X)}{\partial \beta_k} = \sigma_k^\intercal p_{n_X}(X)$ is a linear combination (the coefficients being the $k$th column of $\Sigma$) of the gradient vector of the relative price with respect to the natural logarithm of the production process.

[7] In Appendix D, I establish that $P_0(\omega,t)^2 \frac{\partial p_n(\omega,t)}{\partial \beta_k(\omega,t)} = \int_t^T P_{0,s}(\omega,t)^2 \frac{\partial p_{n,s}(\omega,t)}{\partial \beta_k(\omega,t)} \mathrm{d}s$, which requires $\frac{\partial p_n(\omega,t)}{\partial \beta_k(\omega,t)}$ to have the sign of $\frac{\partial p_{n,s}(\omega,t)}{\partial \beta_k(\omega,t)}$ if the latter derivative maintains the same sign at all $s \in [t,T]$. Yet, as the expectation operator readily commutes inside the time-integrals, signing $\frac{\partial p_{n,s}(\omega,t)}{\partial \beta_k(\omega,t)}$ is nothing but the problem I study in this paper when $s$ is the terminal date. And this sign, being determined solely by the entries of the constant dispersion matrix $\Sigma$, is indeed the same at all $s$. The analysis of Sections 3-5, being built upon identifying the sign of $\frac{\partial p_{n,s}(\omega,t)}{\partial \beta_k(\omega,t)}$, applies at every point of the time interval $[t,T]$; hence, also to the time integral. Time-dependence in the utility flow is not an issue for Sections 3-4 as long as the utility remains CARA or DARA throughout the interval. For Section 5, time-dependence has no bite if the utility remains strictly increasing and concave at all $s$, a redundant requirement really in financial economics.

dividend increases the agent's wealth and, thus, consumption, reducing its marginal utility. Since $P_n(\omega, t)$ is given by the expectation of the product of the dividend with the marginal utility, it need not increase when the dividend increases.[8] Notice also that, albeit each stock's dividend follows a geometric Brownian motion, its equilibrium relative price will not do so apart from a very special case.[9]

The complexity of these dynamics is a binding constraint for economic analysis even when the utility function of the representative agent is such that her optimal portfolio is well-known regarding how it divides her invested wealth between the risky assets and the bond. Suppose, for instance, that she exhibits constant absolute (CARA) or relative (CRRA) risk aversion and that her current invested wealth is \$150 (\$1 representing one unit of consumption) of which \$100 are placed on stocks. Consider also a negative productivity shock that reduces the value of this part of her wealth to \$85.

Other things (in particular, her endowment) being equal, a CARA agent will want to bring the value of the risky part of her portfolio back to \$100, her new optimal split between stocks and bond being \$100/\$35. If the agent exhibits CRRA, on the other hand, she will want to adjust her portfolio so that her invested wealth remains split between stocks and bond in the original 2:1 ratio. She will seek, that is, to invest \$90 in the risky assets and \$45 on the bond.

Since the securities are in fixed supply, their prices must adjust but is not clear how. Obviously, in the CARA case, the prices of at least one stock (since each is in positive net supply) and of the bond (as it is in zero net supply and the agent is risk averse) must rise. But which one is this stock

[8] In Example 1 of Raimondo [34], given $N = K = 1$, log-utility of terminal-date consumption, and no endowment in the terminal period (other than the net supply of the stock), $P_n(\omega, t)$ is actually constant. The adjustment in the relative price, that is needed to clear the markets in the face of the stochastic dividend, obtains entirely through the price of the bond.

[9] Assuming $N = K = 1$ and no terminal-period endowment, Bick [9] established that the relative price of the risky security will follow a geometric Brownian motion in equilibrium if and only if the representative agent's utility over terminal consumption exhibits constant relative risk aversion. For general dimensions of the Brownian and production processes, this has been confirmed by Raimondo [34] (see his Remark 1 and Example 1).

and what happens to the relative prices? Similarly, under CRRA, the price of the bond must again rise whereas that of at least one stock must now fall, along with the corresponding relative price. Which one, though, is this stock as well as the behavior of the other relative prices remain unclear.

The present paper sheds light to the way the equilibrium relative prices respond in the face of such shocks. It does so by analyzing the economic mechanism that determines how the relative prices change (Section 3), and by identifying settings of economic primitives under which these changes can be unambiguously foretold (Section 4). To this end, I begin by establishing how $p_n(\omega, t)$ varies in response to a change in the current realization of the entire Brownian vector $\beta(\omega, t)$ (Theorem 1). It follows immediately from this result that, if the $n$th dividend is correlated with only one Brownian motion, the relative price of the $n$th stock is monotone in the realization of this risk factor. This applies to each and every risky security in the model in two important cases: when there is a single source of risk in the economy ($K = 1$) or if the $N \times K$ dispersion (or factor loadings) matrix $\Sigma$, whose typical row is $\sigma_n^{\intercal}$, is diagonal (in which case, the processes $\beta$ and $Y$ are related injectively).

As the theorem is universally valid regarding the specifications for the agent's utility and her endowment, in both the CARA and CRRA examples above, when $N = K = 1$, the stock's relative price must decrease (increase) if its dividend is positively (negatively) correlated with the underlying Brownian motion. Under general dimensions of the Brownian and production processes, the same holds, in either example, for the relative price of any risky security whose dividend is correlated with only one Brownian motion and the productivity shock originates from this Brownian component alone.

I proceed to study the derivative $\frac{\partial p_n(\omega,t)}{\partial \beta_k(\omega,t)}$ when the $n$th dividend is not correlated with the $k$th Brownian motion ($\sigma_{nk} = 0$). To the extent that this Brownian component does affect some other dividend or the agent's endowment, it induces wealth effects which may require adjustments in the $n$th relative price. Evidently from my analysis (Section 4), the derivative in question will not be zero apart from quite unusual cases. The dynamics of the equilibrium relative prices are in general complex because changes in the underlying Brownian process induce wealth effects which alter not only the agent's risk aversion but also her perceived "riskiness" of a stock. As it turns out, the latter effect is a fundamental mechanism behind the market-clearing induced relative price dynamics. It has been hitherto ignored by the literature, but is examined here in detail (Section 3).

The equilibrium relative price dynamics can be described analytically in some situations. As I show in Section 4, the equilibrium relative price of a stock will typically vary with the realization of a Brownian motion even when its dividend is not correlated with that Brownian component. And this is true even when the agent exhibits CARA. In fact, under CARA, the $n$th relative price and the $k$th Brownian motion are not correlated if and only if this Brownian component and the Brownian motions which are correlated with the $n$th dividend affect the agent's wealth through independent channels. That such separation in the wealth components is sufficient is given by Proposition 3. Necessity, on the other hand, follows from Proposition 2 which identifies settings of economic primitives under which the separation is violated and, even though $\sigma_{nk} = 0$ and the agent exhibits CARA, the $n$th relative price varies (monotonically) with the current realization of $\beta_k$.

With respect to general risk attitudes of the representative agent, Proposition 1 indicates settings under which the relative price of a stock will vary monotonically with a Brownian motion, that is not correlated with its dividend, under any decreasing absolute risk aversion (DARA) utility. One such setting has $\Sigma$ diagonal and the agent's endowment deterministic (Corollary 1.2); admittedly, the most inhospitable economic environment for cross-correlations in prices.

Overall, my analysis shows that, mostly through the asset "riskiness" effect, market-clearing generates correlations across relative asset prices and, hence, returns, over and above those induced by correlations between their respective payoffs. In the model under study, this is a generic phenomenon and the induced correlations are stochastic, even though the covariance coefficients of the dividends are constant. Of course, as I discuss in the next section, the possibility for correlation in asset prices and returns, when there is no common factor in cash flows, is well-known in the literature as "contagion." But it has not been demonstrated before analytically in a general equilibrium model.[10]

The present paper contributes also to the literature on the existence of general equilibrium in continuous-time finance models. In the economy I examine, when the securities' market is potentially dynamically complete ($N = K$), it is in fact dynamically complete if and only if the dispersion matrix $\Sigma$ is nonsingular (Theorem 2). Even though rather widely asserted in the relevant literature, this claim has not been shown explicitly before. More importantly, it is universal with respect to the specification of the representative agent's utility function and her endowment. It applies, therefore, to the given economy even when there are many individuals with heterogenous preferences. As I discuss in detail in the next section, this result patches a hitherto open hole at a critical point in the armor of the literature on the existence of general equilibrium in continuous-time finance models.

The role of dynamic completeness has been important in the literature as a matter of strategy towards proving existence of equilibrium. As a matter of economics, however, it becomes crucial when it comes to pricing financial derivatives. If the pricing process of the underlying securities is dynamically complete, then options and other derivative securities can be uniquely priced by arbitrage arguments and replicated by trading the underlying securities. In the absence of dynamic completeness, however, this is no longer the case; arbitrage considerations do not suffice to determine unique option prices and replication is not possible.

[10]The literature on contagion has focused mostly on the propagation of shocks across national or regional stock markets. One of its peculiarities is that, although there is fairly widespread agreement about the contagion events themselves, there is no consensus on exactly what constitutes contagion or how it should be defined. One preferred definition is the propagation of shocks in excess of that which can be explained by fundamentals. Another (often referred to as shift-contagion) classifies contagion as a change in how shocks are propagated between normal and crisis periods. Yet another labels contagion the transmission of shocks through specific channels, such as herding or irrational investor behavior. And an even broader definition identifies contagion as any linkage mechanism that causes markets or asset prices to move together. The main reason for this prolificness is that each definition seems to run in its own difficulties when it comes to empirical identification. My focus being strictly theoretical in the present paper, I will be referring to contagion having in mind the first definition.

Given a financial environment, therefore, it is fundamental to be able to associate dynamic completeness with at least some of its economic primitives in a manner that remains unambiguously verifiable and holds generically across the space of these primitives. This is precisely the contribution of Theorem 2 with respect to the securities market and the economy under study.

The remaining of the paper is organized as follows. In the next section, the model I study and the results I obtain throughout the paper are placed in the context of the pertinent literature. Section 3 investigates the comparative statics of the equilibrium relative price of the typical stock with respect to the typical Brownian motion, its emphasis being on economic intuition and interpretation. In Section 4, I take this investigation further aiming at specific claims regarding these relative price dynamics. Section 5 presents the result on dynamic completeness while Section 6 concludes. All proofs, as well as some supporting technical material, can be found in the Appendix.

## 2 Related Literature

The theoretical backdrop of the relative price dynamics investigated by the present paper has been the subject of a number of well-known studies in continuous-time general equilibrium asset-pricing. To name but a few, in Bick [9], Raimondo [34], as well as Anderson and Raimondo [6], the production, consumption, information, trading, and preferences structures but also the dividends' specification are exactly as described in the previous section.[11] The models of Bick [8] as well as He and Leland [21], which have

[11] The consumption and trading structures in Bick [9] differ slightly from the ones I presented but these discrepancies bear no effect on the equilibrium prices. The author restricts attention to a dynamically-complete securities' market with $N = K = 1$. His equilibrium being essentially an Arrow-Debreu one, it suffices that the assets are traded only once, at $t = 0$. Raimondo [34] as well as Anderson and Raimondo [6], on the other hand, do not restrict the dimensionality of the Brownian and production processes. Since their securities' market can be also dynamically incomplete, their securities have to be traded continuously. These papers differ only in the specification of the terminal dividends: Anderson and Raimondo [6] (and Bick [9] for that matter) consider the general

no real differences between them, are also essentially the same as the one I have outlined.[12] Both papers describe the assets in terms of their pricing rather than their dividend processes. Yet, these pricing processes are given in units of consumption so that holding an asset is equivalent to receiving its price as consumption dividend. Either paper assumes $N = K = 1$ and that the representative agent has no endowment, other than the net supply of the stock (which can be viewed as the market portfolio). These two restrictions are also present in Bick [9]. As a consequence, in all three papers consumption takes place only at the final date. By contrast, Raimondo [34] as well as Anderson and Raimondo [6] assume that the agent is endowed with a deterministic flow rate of consumption during the interval $[0, T)$ and with a lump sum at $T$, which may be stochastic (a continuous function of the terminal-date realization of the underlying Brownian process).

The aforementioned papers examine the issue of existence of general equilibrium asset prices under two different approaches. Raimondo [34] as well as Anderson and Raimondo [6] construct the equilibrium pricing process directly from the economic primitives, in a manner that is valid when the securities' market is potentially dynamically complete ($N = K$) but also when it is necessarily dynamically incomplete ($N < K$). The remaining papers restrict attention to the securities' market being in fact dynamically complete. They assume that the stock price follows a given diffusion process and proceed to identify necessary and sufficient conditions for it to be an equilibrium pricing process. Despite the two perspectives, however, all papers share essentially the same underlying economic structure, the one outlined in the introductory section. More importantly for the purposes of my study, in all of them the equilibrium relative price of the typical stock is given by (1).[13]

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geometric Brownian motion whereas Raimondo [34] its special case in which $\mu_n = 0$ for all $n$ and $\Sigma = \mathrm{I}_K$.

[12] As opposed to the trading structure of Bick [9] (and in agreement to the one I assumed in the introductory section), the securities in Bick [8] as well as in He and Leland [21] are traded continuously.

[13] I am referring to Theorems 1 and 2.1 of Raimondo [34] and Anderson and Raimondo [6], respectively. As I have done, both papers take consumption as the numeraire. In Bick [9], see equation (4) and the very intuitive argument for why it is a necessary equilibrium

Even though all of these papers are important and well-known, probably the most seminal study of the finite-horizon, continuous-time, single-good economy with identical agents and Lucas trees is Cox et al. [13], an extension of the setting in Lucas [27]. Lucas considered an infinite-horizon, discrete-time, single- and perishable-good, pure-exchange economy with several trees in which a representative agent with state- and time-independent utility for instantaneous consumption and no endowment (other than the trees) has continuous access to the trees' output, so that intermediate consumption is financed by the trees' dividends. Cox and his co-authors present the continuous-time analogue of Lucas' model, enhancing it to include production.

As in the previously-mentioned papers, an underlying stochastic process generates shocks to the productivity of the trees. Yet, the trees' productivity is now influenced also by the representative agent who has continuous access to the trees' output, consuming some and reinvesting the rest in the production process. Cox et al. [13] consider in addition a more general preference structure but rather a more restricted trading one. The agent may have now state- and time-dependent preferences for instantaneous consumption while there is a dynamically complete securities market in which a full set of Arrow-Debreu contingent claims are traded (each available in zero net supply).

Allowing for time- but not state-dependence, the representative agent of Cox et al. [13] seeks to maximize the current expectation of the entire

relation, bearing in mind that this author chose the bond as the numeraire. Hence, $P_0(\omega, t) = 1$ at all $(\omega, t) \in \Omega \times [0, T]$ and, given that consumption occurs only at the final date, $P_c(\omega, t) = \mathbb{E}[u'(W(\omega, T)) | \mathcal{F}_t]$. The same pricing equation supports also the analysis of Bick [8], which characterizes general diffusions as equilibrium price processes. In fact, Bick makes here explicit reference (in the proof to the corollary that follows Proposition 1) to equation (4) of his earlier paper. By contrast, He and Leland [21] characterize general diffusions as equilibrium pricing processes by identifying necessary and sufficient conditions for the appropriate partial differential equations. Their approach does not involve conditional expectations of the marginal utility of consumption. Yet, as established by their Corollary 1, their analysis and Bick's are in complete agreement when the stock prices (which are given in units consumption and, thus, coincide with the dividends) are restricted to be time-homogenous diffusions, a family of processes of which the geometric Brownian motion is a member.

future utility flow, $\mathbb{E}_\pi\left[\int_t^T u\left(W\left(\mathcal{I}\left(\omega,s\right)\right),s\right)\mathrm{d}s|\mathcal{F}_t\right]$. In this case, the equilibrium price of any real asset relative to that of the zero-coupon bond is given by (2).[14] The same pricing formula can be found also in Cochrane et al. [12] (see Equation 20), Martin [28], Duffie and Zame [15] (see Theorem 1 and the subsequent discussion in Section 5), Karatzas et al. [22] (Corollary 10.4), Riedel [35] (Theorem 2.1), and Wang [38] (Equation 2.4).[15] Notice finally that, even when the individuals in the economy have non-identical preferences for consumption, the pricing formula takes still the same basic form as in (1)-(2). The only difference is that the individual marginal utilities are now taken at the equilibrium consumptions of the agents, which are determined endogenously as part of the equilibrium (see, for instance, Duffie and Zame [15] or Anderson and Raimondo [5]).

Regarding the study of equilibrium relative price dynamics per se, the works that are closest to the present are Cochrane et al. [12] and Martin [28]. The latter being a generalization of the former, both papers investigate special cases of the pure-exchange infinite-horizon version of the economy in Cox et al. [13]. Cochrane and his co-authors consider a representative agent with log-utility for instantaneous consumption who has access to the dividend stream of two Lucas trees, each following a geometric Brownian motion (or being constant if the tree represents a zero-coupon bond). The authors characterize the asset-price and return dynamics that result from market-clearing in this context. They obtain closed-form solutions for a large collection of variables of interest such as absolute prices, expected returns,

[14] I am referring to the last term of equation (38) in Cox et al. [13] (whose notation pretermits the dependence upon $\Omega$). This term prices real assets, claims that pay $\delta\left(W\left(s\right),Y\left(s\right),s\right)$ units of consumption at time $s$ when the realization of the stochastic process is $Y\left(s\right)$ (the zero-coupon bond, for instance, has $\delta\left(Y\left(s\right),s\right)=1$ at all $s$). By contrast, the first two terms in (38) allow for the pricing of general financial assets, including options and futures. More precisely, claims that pay $\Theta\left(W\left(T\right),Y\left(T\right)\right)$ if some underlying variables do not leave a certain region before the maturity date $T$ and $\Psi\left(W\left(s\right),Y\left(s\right),s\right)$ every time $s$ they do, otherwise. Notice that $J\left(W\left(s\right),Y\left(s\right),s\right)$ is the agent's equilibrium indirect utility at time $s$, given the realization $Y\left(s\right)$. It depends on the date $s$ and the state variable $Y$ as the authors allow for the direct utility to be time- and state-dependent. As I establish in Appendix D, all of my results remain valid in the face of the former dependence. The latter is a level of generality beyond the scope of my study.

[15] Wang's pricing formula derives actually from a particular case of the analysis in Duffie and Skiadas [14] (Example 3).

market-betas, and return-correlations. Yet, these are given with respect to the dividend-share (the share of total output due to a tree's dividend) rather than the underlying risk process, while the corresponding dynamics are examined numerically rather than analytically.

The solution method in Cochrane et al. [12] depends fundamentally upon the dividend-share being the unique state variable, in a way that makes it applicable only to log-utility and two trees. By contrast, Martin [28] uses an approach that extends to power utility and many trees, whose dividend streams may follow geometric Brownian motions with (normally-distributed) jumps, offering also closed-form solutions for absolute prices, expected returns, and bond-yields. However, these solutions are given in terms of a state-vector which is not the underlying stochastic process (it depicts instead the relative sizes of the dividends), while the corresponding dynamics are presented again through calibrations.

Both papers draw a substantial part of the reader's attention to the fact that there is significant price comovement even between assets whose dividends are independent. The intuition is somewhat clear in the case of two trees ($N = K \leq 2$ in my notation). When one asset has a positive dividend shock, other things being equal, its dividend becomes a larger share of a now larger total consumption. As a result, investors want to rebalance by spreading some of their larger wealth across both trees. In the face of the fixed net supply, though, they cannot collectively rebalance, so asset prices must adjust.

Typically, the price of the tree with the positive shock rises whereas the risk premium of the other falls. If the two dividend streams are independent, given no shock on the second dividend, its risk premium can fall only via an increase in its price. Given no news about its own cash flow, the fact that it now constitutes a smaller part of total consumption typically means that the asset becomes less positively correlated with consumption. Ergo, investors want to hold more of the second asset but cannot, forcing instead its price to rise.

But this is what happens typically, not always, because the relation between an asset's risk-premium and the dividend-share does not depend only on this “cash-flow beta” intuition. It depends also on “valuation-beta,” the tendency of the price-dividend ratio to change with the market and, thus, total consumption. And the latter relation is not always positive; there are ranges of dividend-share values where the price of the second asset falls in the preceding example (see Figure 3 in Cochrane et al. [12] and Figure 7(a) in Martin [28]). This is most evident when the second asset is a zero-coupon bond ($N = K = 1$). Given its smaller dividend-share, it is still true that investors want to spread their larger wealth across both trees, which should raise the price of the bond. Yet, the interest rate also changes, and this may more than offset the rebalancing desire (see Figure 9 in Cochrane et al. [12]).

As shown by my analysis, however, the ambiguous nature of the asset-price dynamics in the above example is mostly due to the variable with respect to which these are examined by the two papers. Be it the dividend-share or the relative size of the dividends, the evolution of the state-variable depends, in either paper, on that of the underlying stochastic process in a way that is not clear unless $N = K = 1$. Both papers attempt in effect to relate a change in the current realization of one of the dimensions of the underlying stochastic process to asset-prices via a state-variable whose own change cannot be isolated to come from that dimension alone. In the present paper, by contrast, I study the asset-price dynamics with respect to the underlying stochastic process directly. As it turns out, there are settings of economic primitives under which these dynamics are not ambiguous at all. In fact, in either of the above examples, they are described by Theorem 1 and Corollary 1.2, analytically and for any DARA utility.

Of course, the deployment of an intermediate state-variable allows for calibrations that show to what extent asset-price comovements are quantitatively important. Nevertheless, when the goal is purely theoretical, to understand the economic dynamics induced by market-clearing, this comes at the cost of obscuring the distinction between two separate channels through which shocks to current wealth affect asset prices: by changing the agent’s risk aversion but also by altering her perception of the “riskiness” of a security. The dynamics of the former mechanism are well-known and straightforward. Those of the latter have not, to the best of my knowledge, hitherto been analyzed by the finance literature and are complex.

As shown in the next section but also by Corollary 1.2, under DARA and independent dividend streams, the two mechanisms operate in the same direction, leading to positive contemporaneous correlation in relative asset prices. But this is by no means universally the case. The operation of the asset-riskiness effect on relative prices can be isolated under CARA since the risk-aversion channel leaves then relative prices unchanged. As attested by Proposition 2 or Corollary 2.1, it can lead to negative correlation.

The possibility for a "common factor" or "contagion" in asset prices (and, thus, returns) to emerge, when there is no common factor in cash flows, is well-known but has not been demonstrated before analytically in a general equilibrium model. It is noted, for example, in Raimondo [34] as well as Anderson and Raimondo [6] but no formula is given for the cross-derivative. Kodres and Pritsker [23], Kyle and Xiong [24], but also Lagunoff and Schreft [25] show that contagion can obtain as a wealth effect in rational expectations equilibria. These are not general equilibrium models, however, as some market participants are not rational (the former two models require the presence of noise traders, the latter of irrational ones). Contagion equilibria arise as well in Aliprantis et al. [1] within the context of a monetary model where players engage, though, in strategic, non price-taking behavior.

On the empirical side, the literature has focused mostly on contagion across national or regional stock markets (see, for instance, Shiller [37] or Forbes and Rigobon [17]). Yet, to name but a couple of studies, Gropp and Moerman [19] identify within-country contagion among large European bank stocks while Pindyck and Rotemberg [33] find evidence of excess correlation in asset price comovements. There is also ample evidence that conditional correlations across asset prices and returns are stochastic, and of a magnitude that cannot be explained by covariances between their respective payoffs alone.[16] Both, phenomena that my analysis finds to be generic and

[16]In a seminal study, Fama and French [16] identified a set of common risk factors that explained the expected returns on stocks and bonds. Similarly but more recently, Moskowitz [32] found evidence that risk-premia are better represented by covariances

due to market-clearing alone, since the assumed covariances between asset payoffs are constant.

To conclude relating the present paper to the pertinent literature, Theorem 2 should be viewed in the context of the results on existence of general equilibrium in continuous-time finance models, established by a number of important papers (some mentioned already, others including Duffie and Skiadas [14] as well as Anderson and Raimondo [5]). To the best of my knowledge, apart from Raimondo [34] or Anderson and Raimondo [6], all of these papers are supposed to operate within the realm of a potentially dynamically complete securities market ($N = K$). Yet, none of them specifies explicitly the economic primitives under which this condition will be met.

The typical approach has been to start with a given candidate equilibrium price process, which is assumed to be dynamically complete, and proceed to establish that it is in fact an equilibrium.[17] However, the candidate equilibrium price processes are determined from the economic primitives of the model (the utility functions of the agents, their endowments, and the dividend processes of the securities) by a fixed point argument. And this means that, except in the extremely special cases where one can solve for the candidate equilibrium explicitly, it is not possible to verify from the primitives if the candidate equilibrium price process is indeed dynamically complete.

By contrast, Raimondo [34] as well as Anderson and Raimondo [6] allow for the case when the market is necessarily dynamically incomplete ($N < K$). The latter being a direct extension of the former, both papers study a representative-agent economy and manage to construct the appropriate Negishi weights (hence, the equilibrium pricing process) directly from its primitives. Neither, however, specifies when the market is in fact dynamically complete, given that it is potentially so, while following their

with the implied market portfolio than by own-variances. Andersen and Lund [4], on the other hand, suggest that U.S. risk-free short-term interest rates can be consistently estimated as stochastic-volatility diffusions. On stochastic second moments of returns, see also Andersen et al. [3]-[2], Longin and Solnik [26] or Schwert and Seguin [36].

[17] Of course, the form of the assumption varies in the literature. See the introductory section of Anderson and Raimondo [5] for a summary review and discussion.

arguments is quite demanding as nonstandard analysis is heavily used.

Similar economic intuition and mathematical apparatus is deployed also in Anderson and Raimondo [5]. Generalizing their previous work to an economy with many heterogeneous agents, the authors introduce here a condition on the economic primitives (in particular, on the dividends of the securities) which guarantees dynamic completeness, permitting the construction of the equilibrium pricing process via their representative agent approach. As shown in Section 5, their economy embeds the one I examine in the present paper as a special case in which their condition reduces to the dispersion matrix $\Sigma$ being nonsingular.

In this sense, Theorem 2 verifies that indeed the condition in Anderson and Raimondo [5] suffices for dynamic completeness (using, though, completely standard mathematical apparatus). More importantly perhaps, it renders the condition also necessary for dynamic completeness, even with many heterogenous agents. This is because my result applies for any specification of the representative agent's utility function or her endowment, while dynamic completeness is a primitive economic property, independent of the agents' preferences and endowments. If the securities market is dynamically complete, the equilibrium asset-pricing process must agree with the pricing process of some representative agent. And whatever her preferences, Theorem 2 requires that $\Sigma$ is nonsingular.

# 3 Mechanics of Comparative Statics

Even though my analysis applies also for the pricing relation in (2), my exposition will refer to the one in (1), which can be supported by the following theoretical foundation. For any $\omega \in \Omega$, the dividends of the $N+1$ securities are zero at intermediate dates $t \in [0, T)$ but $D_0(\mathcal{I}(\omega, T)) = 1$ and $D_n(\mathcal{I}(\omega, T)) = e^{\mu_n T + \sigma_n^\intercal \beta(\omega, T)}$ for $n = 1, \ldots, N$ at the end. The representative agent's endowment process is deterministic, except possibly at $T$, when it is given by $\rho(\mathcal{I}(\omega, T))$ for some continuous function $\rho : \mathbb{R}^K \times \{T\} \to \mathbb{R}_+$. The agent's wealth (equivalently, her equilibrium consumption) equals, therefore, her deterministic endowment during the intermediate period and

$$W(\omega, T) = \rho(\mathcal{I}(\omega, T)) + \sum_{n=1}^{N} D_n(\mathcal{I}(\omega, T))$$

at the end. She also has an additively-separable, time-independent utility function which, for a measurable with respect to the Brownian filtration consumption function $c : \Omega \times [0, T] \to \mathbb{R}_{++}$, is given by

$$U(c(\mathcal{I}(\omega, t))) = \mathbb{E}_\pi \left[ \int_t^T v(c(\mathcal{I}(\omega, s))) \, \mathrm{d}s + u(c(\mathcal{I}(\omega, T))) \, | \mathcal{F}_t \right] \tag{3}$$

for some instantaneous utility functions $v, u : \mathbb{R}_{++} \to \mathbb{R}$ that are everywhere twice continuously-differentiable, strictly increasing, and strictly concave.

The corresponding equilibrium pricing process has been derived explicitly by Raimondo [34], in terms of the agent's utility function, her terminal-period endowment, and the current realization $\beta(\omega, t)$ of the Brownian vector:[18]

$$\begin{aligned}
P_n(\omega, t) &= \mathbb{E}_\pi \left[ u'(W(\mathcal{I}(\omega, T))) \, D_n(\mathcal{I}(\omega, T)) \, | \mathcal{F}_t \right] \\
&= \int_{\mathbb{R}^K} u'(W(\mathcal{I}(\omega, t), \mathbf{x})) \, e^{\mu_n T + \sigma_n^\intercal \left(\beta(\omega, t) + \sqrt{T-t}\mathbf{x}\right)} \mathrm{d}\Phi(\mathbf{x}) \\
P_0(\omega, t) &= \mathbb{E}_\pi \left[ u'(W(\mathcal{I}(\omega, T))) \, | \mathcal{F}_t \right] = \int_{\mathbb{R}^K} u'(W(\mathcal{I}(\omega, t), \mathbf{x})) \, \mathrm{d}\Phi(\mathbf{x})
\end{aligned}$$

[18]See Theorem 1 in Raimondo [34] but also Theorem 2.1 in Anderson and Raimondo [6]. All prices are stochastic processes; more precisely, continuous, square-integrable martingales with respect to the Brownian filtration. To obtain the former theorem, Raimondo imposes three additional assumptions. Specifically, the utility functions are bounded below: $\exists K > -\infty$ s.t. $v(c), u(c) > K \; \forall c \in \mathbb{R}_{++}$. Moreover, in order to not have to handle genericity considerations on existence, a short-sale constraint is introduced: $\exists M > 0$ s.t. the agent is not permitted to hold less than $-M$ units of any of the $N + 1$ traded assets. Finally, the terminal-period endowment function is taken to satisfy $0 \leq \rho(\mathbf{x}) \leq r + e^{r|\mathbf{x}|}$ for some $r \in \mathbb{R}_+$ and $\forall \mathbf{x} \in \mathbb{R}^K$. Yet, Anderson and Raimondo [5] show that the first two assumptions are not necessary for the existence of equilibrium. As for the third condition, it is satisfied by any bounded-above function $\rho(\cdot)$. It should be pointed out also that my results per se do not depend upon any assumptions other than the ones already stated in the text. Additional conditions, that may be necessary for an existence proof, are not really relevant for a comparative statics analysis. If an equilibrium price process does indeed exist, the equilibrium relative prices have to be as in (1), and this is where I begin.

Here, the quantities

$$
\begin{aligned}
W(\mathcal{I}(\omega,t),\mathbf{x}) &= \rho\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)+\sum_{n=1}^{N} D_n(\mathcal{I}(\omega,t),\mathbf{x}) \\
D_n(\mathcal{I}(\omega,t),\mathbf{x}) &= e^{\mu_n T+\sigma_n^{\intercal}\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)} \qquad (4)
\end{aligned}
$$

depict, respectively, the terminal realizations of the agent's wealth and of the $n$th dividend, conditional on the current Brownian realization and on its future increment $\beta(\omega,T)-\beta(\omega,t)=\sqrt{T-t}\mathbf{x}$, with $\mathbf{x}\sim\mathcal{N}(\mathbf{0},\mathrm{I}_K)$ and $\Phi(\cdot)$ being the standard $K$-dimensional Normal cumulative distribution function.

Notice that both of the last two quantities above as well as all expectations henceforth are $\mathcal{F}_t$-conditional. It should be kept in mind also that, since the remainder of my analysis applies at all states, the dependence upon $\Omega$ will be pushed aside in the interest of parsimonious notation. My focus will be on the comparative statics of the typical relative price

$$
p_n(t)=\frac{P_n(t)}{P_0(t)}=\frac{\mathbb{E}_{\mathbf{x}}\left[u'(W(\mathcal{I}(t),\mathbf{x}))\,D_n(\mathcal{I}(t),\mathbf{x})\right]}{\mathbb{E}_{\mathbf{x}}\left[u'(W(\mathcal{I}(t),\mathbf{x}))\right]}
$$

with respect to changes in $\beta_k(t)$, the current realization of the typical Brownian motion. As it turns out, the corresponding dynamics are quite complex, surprisingly so in some situations. This section attests to their richness by means of describing the constituent parts of their generating mechanism.

Towards an overview of this mechanism, let us begin by observing that the typical relative price can be expressed also as follows

$$
\begin{aligned}
p_n(t) &= \mathbb{E}_{\mathbf{x}}\left[D_n(\mathcal{I}(t),\mathbf{x})\right]+\frac{\mathrm{Cov}_{\mathbf{x}}\left[u'(W(\mathcal{I}(t),\mathbf{x})),D_n(\mathcal{I}(t),\mathbf{x})\right]}{P_0(t)} \qquad (5) \\
&= \mathbb{E}_{\mathbf{x}}\left[D_n(\mathcal{I}(t),\mathbf{x})\right]+\frac{\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathcal{I}(t),\mathbf{x}+\sqrt{T-t}\sigma_n\right)\right)\right]}{\mathbb{E}_{\mathbf{x}}\left[u'(W(\mathcal{I}(t),\mathbf{x}))\right]} \qquad (6)
\end{aligned}
$$

whereas the second equality follows by Lemma A.2 of Appendix A. As a consequence, we have

$$\frac{\partial P_n(t)}{\partial \beta_k(t)} = \frac{\partial \mathrm{Cov}_{\mathbf{x}}\left[u'\left(W\left(\mathcal{I}(t),\mathbf{x}\right)\right), D_n\left(\mathcal{I}(t),\mathbf{x}\right)\right]}{\partial \beta_k(t)} \tag{7}$$
$$+\mathbb{E}_{\mathbf{x}}\left[D_n\left(\mathcal{I}(t),\mathbf{x}\right)\right]\frac{\partial P_0(t)}{\partial \beta_k(t)} + \sigma_{jk}\mathbb{E}_{\mathbf{x}}\left[D_n\left(\mathcal{I}(t),\mathbf{x}\right)\right]P_0(t)$$

while, on the other hand,

$$\frac{\partial P_0(t)}{\partial \beta_k(t)} = \mathbb{E}_{\mathbf{x}}\left[u''\left(W\left(\mathcal{I}(t),\mathbf{x}\right)\right)\frac{\partial W\left(\mathcal{I}(t),\mathbf{x}\right)}{\partial \beta_k(t)}\right] \tag{8}$$

In words, these equations depict the following relations. Given an arbitrary realization $\beta(t)$ of the underlying stochastic process, exchanging one unit of the bond for one unit of the stock increases the currently (i.e. $\mathcal{F}_t$-conditional) expected terminal-period wealth by the currently expected terminal dividend of the security, $\mathbb{E}_{\mathbf{x}}\left[D_n\left(\mathcal{I}(t),\mathbf{x}\right)|\mathcal{F}_t\right] = e^{\mu_n T + \sigma_n^{\intercal}\left(\beta(t) + \frac{(T-t)\sigma_n}{2}\right)}$. The latter quantity gives the number of bond units one unit of the $n$th stock is equivalent to in terms of terminal-period wealth. In terms of marginal utility (which is what matters in general equilibrium pricing), however, the corresponding equivalence requires also that any realization $\sqrt{T-t}\mathbf{x} \sim \mathcal{N}\left(0,(T-t)\,\mathrm{I}_K\right)$ of the future increment $\beta(T) - \beta(t)$ gets translated by the quantity $(T-t)\,\sigma_n$.

### The Own-Dividend Effect

Other things remaining equal, a change $\mathrm{d}\beta_k(t)$ in the $k$th component of $\beta(t)$ alters by $\sigma_{nk}\mathrm{d}\beta_k(t)$ the $\mathcal{F}_t$-conditional drift, $\mu_n T + \sigma_n^{\intercal}\beta(t)$, of the underlying stochastic process that determines the $n$th terminal dividend.[19] The $\mathcal{F}_t$-conditional expectation of the terminal dividend itself, then, changes by $\sigma_{nk}\mathbb{E}_{\mathbf{x}}\left[D_n\left(\mathcal{I}(t),\mathbf{x}\right)\right]\mathrm{d}\beta_k(t)$. Suppose now that $\beta_k(t)$ increases. If $\sigma_{nk} > 0$ ($\sigma_{nk} < 0$), the currently expected terminal dividend will be higher (lower). Due to non-satiation ($u'(\cdot) > 0$), this increases (decreases) the willingness of the agent to hold the $n$th risky security. As she must, though, continue to

[19] "Other things remaining equal" (or similar expressions) refer henceforth to the current realizations of the remaining $K-1$ sources of uncertainty, $\{\beta_m(t)\}_{m\in\{1,\ldots,K\}\setminus\{k\}}$.

hold its net supply in equilibrium, the (absolute) price of the security must rise (fall) exactly by $\sigma_{nk} P_0(t) \mathbb{E}_{\mathbf{x}} [D_n(\mathcal{I}(t), \mathbf{x})] \mathrm{d}\beta_k(t)$, which is the change in the $\mathcal{F}_t$-conditional drift of the underlying stochastic process in units of the bond. Henceforth, I will be referring to this as the *own-dividend effect* of $\mathrm{d}\beta_k(t)$ on the $n$th equilibrium price. It is depicted by the third term on the right-hand side of (7).

**The Wealth Effect**

For any future realization $\sqrt{T-t}\mathbf{x}$ of the stochastic process $\beta(T) - \beta(t)$, a change in $\beta_k(t)$ corresponds to revealing information that changes also the $\mathcal{F}_t$-conditional expected terminal dividend of any security $n' \in \{1, \ldots, N\}$ by $\sigma_{n'k} \mathbb{E}_{\mathbf{x}} [D_{n'}(\mathcal{I}(t), \mathbf{x})] \mathrm{d}\beta_k(t)$. These changes along with that in the terminal-period endowment, $\mathrm{d}\rho\left(\beta(t) + \sqrt{T-t}\mathbf{x}\right)$, give the corresponding change in the $\mathcal{F}_t$-conditional terminal-period wealth. Ceteris paribus, the agent's risk aversion ($u''(\cdot) < 0$) induces an opposite change in marginal utility, termed from now on the *wealth effect* of $\mathrm{d}\beta_k(t)$.

Regarding the equilibrium price of the bond, this effect is given by equation (8). With respect to the equilibrium price of the $n$th risky security, it is given by the second term on the right-hand side of (7). Clearly, the direction of the wealth effect is the same on either price. In fact, this is true also for its magnitude since the two terms differ only by the proportionality constant needed to convert units of the stock into units of the bond, in terms of $\mathcal{F}_t$-conditional expected terminal-period wealth.

To identify the effect on the $n$th relative equilibrium price, consider its derivative

$$\frac{\partial p_n(t)}{\partial \beta_k(t)} = \frac{1}{P_0(t)} \left[ \frac{\partial P_n(t)}{\partial \beta_k(t)} - p_n(t) \frac{\partial P_0(t)}{\partial \beta_k(t)} \right] \tag{9}$$

Using equations (8) and (6) and the second term on the right-hand side of

(7), it is straightforward to verify that the wealth effect is given by

$$\begin{aligned}
& \left(\frac{\mathbb{E}_{\mathbf{x}}\left[D_n\left(\mathcal{I}\left(t\right),\mathbf{x}\right)\right]-p_n\left(t\right)}{P_0\left(t\right)}\right)\frac{\partial P_0\left(t\right)}{\partial\beta_k\left(t\right)} \qquad (10)\\
= \; & \mathbb{E}_{\mathbf{x}}\left[D_n\left(\mathcal{I}\left(t\right),\mathbf{x}\right)\right]\left(1-\frac{\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathcal{I}\left(t\right),\mathbf{x}+\sqrt{T-t}\sigma_n\right)\right)\right]}{\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathcal{I}\left(t\right),\mathbf{x}\right)\right)\right]}\right)\frac{\frac{\partial P_0(t)}{\partial\beta_k(t)}}{P_0\left(t\right)}
\end{aligned}$$

**The Asset-Riskiness Effect**

Given $\sqrt{T-t}\mathbf{x}$, the extent to which $\mathrm{d}\beta_k\left(t\right)$ alters $P_n\left(t\right)$ by changing the marginal utility of terminal-period wealth depends on the future realization of the $n$th terminal dividend. Similarly, the extent to which $\mathrm{d}\beta_k\left(t\right)$ alters $P_n\left(t\right)$ via a change in the $n$th terminal dividend depends on the future realization of the marginal utility of terminal-period wealth. Which is to say that changes in $\beta_k\left(t\right)$ affect the equilibrium price of the $n$th risky security through changes in the correlation between the marginal utility of terminal-period wealth and the terminal dividend of the security. I will be referring to this as the *asset-riskiness effect* of $\mathrm{d}\beta_k\left(t\right)$ on $P_n\left(t\right)$. It is depicted by the first term on the right-hand side of equation (7).

To understand the mechanics of this effect, it is instructive to consider a setting in which (i) the components of the Brownian process that are correlated with the $n$th dividend ($\beta_m\left(t\right)$ with $\sigma_{nm}\neq 0$) affect the terminal-period wealth only through this dividend, and (ii) the $k$th Brownian component is not correlated with the $n$th dividend ($\sigma_{nk}=0$). Formally, let

$$K_n=\left\{m\in\left\{1,\ldots,K\right\}:\sigma_{nm}\neq 0\right\}$$

be the collection of the Brownian components that affect $D_n\left(t\right)$. Suppose also that $k\notin K_n$ and consider the terminal-period wealth specification

$$\begin{aligned}
W\left(\mathcal{I}\left(t\right),\mathbf{x}\right) &= \rho\left(\mathcal{I}\left(t\right),\mathbf{y}\right)+\sum_{n'\notin K_n}D_{n'}\left(\mathcal{I}\left(t\right),\mathbf{y}\right)+D_n\left(t,\mathbf{z}\right)\\
&\equiv W_{-M}\left(\mathcal{I}\left(t\right),\mathbf{y}\right)+D_n\left(t,\mathbf{z}\right) \qquad (11)
\end{aligned}$$

where $M = |K_n| < K$ and $\mathbf{x} = (\mathbf{z},\mathbf{y}) \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} \mathrm{I}_M & \mathbb{O}_{M\times(K-M)} \\ \mathbb{O}^{\intercal} & \mathrm{I}_{K-M} \end{bmatrix}\right)$ (with $|\cdot|$ and $\mathbb{O}$ denoting, respectively, the cardinality of a set and the zero matrix). In this case, $\frac{\partial W(\mathcal{I}(t),\mathbf{x})}{\partial\beta_k(t)} = \frac{\partial W_{-M}(\mathcal{I}(t),\mathbf{y})}{\partial\beta_k(t)}$ so that the first term on the right-hand side of (7) can be written out as follows

$$\begin{aligned} & \mathrm{Cov}_{\mathbf{x}}\left[u''\left(W\left(\mathcal{I}\left(t\right),\left(\mathbf{z},\mathbf{y}\right)\right)\right)\frac{\partial W\left(\mathcal{I}\left(t\right),\mathbf{y}\right)}{\partial\beta_k\left(t\right)}, D_n\left(t,\mathbf{z}\right)\right] \\ = & \int_{\mathbb{R}^{K-M}}\begin{pmatrix} \int_{\mathbb{R}^M} u''\left(W\left(\mathcal{I}\left(t\right),\left(\mathbf{z},\mathbf{y}\right)\right)\right)D_n\left(t,\mathbf{z}\right)\mathrm{d}\Phi\left(\mathbf{z}\right) - \\ \int_{\mathbb{R}^M} u''\left(W\left(\mathcal{I}\left(t\right),\left(\mathbf{z},\mathbf{y}\right)\right)\right)\mathrm{d}\Phi\left(\mathbf{z}\right)\int_{\mathbb{R}^M} D_n\left(t,\mathbf{z}\right)\mathrm{d}\Phi\left(\mathbf{z}\right) \end{pmatrix} \\ & \times\frac{\partial W_{-M}\left(t,\mathbf{y}\right)}{\partial\beta_k\left(t\right)}\mathrm{d}\Phi\left(\mathbf{y}\right) \\ = & \int_{\mathbb{R}^{K-M}}\mathrm{Cov}_{\mathbf{z}}\left[u''\left(W\left(t,\left(\mathbf{z},\mathbf{y}\right)\right)\right), D_n\left(t,\mathbf{z}\right)\right]\frac{\partial W_{-M}\left(t,\mathbf{y}\right)}{\partial\beta_k\left(t\right)}\mathrm{d}\Phi\left(\mathbf{y}\right) \quad (12) \end{aligned}$$

In this setting, conditional on the realization $\mathbf{y}$, the terminal-period wealth $W(\mathcal{I}(t),\mathbf{x})$ is strictly comonotonic in $\mathbf{z}$ with $D_n(t,\mathbf{z})$. Under non-increasing absolute risk aversion (NARA), so is $u''(W(\mathcal{I}(t),\mathbf{x}))$ which implies, in turn, that the covariance within the integral above is strictly positive (see Appendix B).[20] Clearly, the sign of the asset-riskiness effect of $\mathrm{d}\beta_k(t)$ on $P_n(t)$ will be given by the sign of $\frac{\partial W_{-M}(\mathcal{I}(t),\mathbf{y})}{\partial\beta_k(t)}$, as long as the latter remains unchanged on $\mathbb{R}^{K-M}$.

Recall, however, that the wealth effect of $\mathrm{d}\beta_k(t)$ on $P_n(t)$ obtains always in the same direction as the wealth effect on $P_0(t)$. And $\frac{\partial P_0(t)}{\partial\beta_k(t)}$ is required by (8) to have the opposite sign of $\frac{\partial W_{-M}(\mathcal{I}(t),\mathbf{y})}{\partial\beta_k(t)}$. In this setting, therefore, the asset-risknes and wealth effects push $P_n(t)$ in opposite directions under NARA. The intuition why is straightforward. Let, for instance, $\frac{\partial W_{-M}(\mathcal{I}(t),\mathbf{y})}{\partial\beta_k(t)} > 0\ \forall\mathbf{y}\in\mathbb{R}^{K-M}$. An increase in $\beta_k(t)$ raises the $\mathcal{F}_t$-conditional terminal-period wealth, reducing its marginal utility. Under NARA, though, the decrease in $u'(W(\mathcal{I}(t),\mathbf{x}))$ is smaller when $D_n(t,\mathbf{z})$ is large and larger when it is small. Which, due to risk aversion, means that the increase in $\beta_k(t)$ makes the terminal-period wealth less positively correlated with the

[20] The coefficient of absolute risk-aversion is the function $r_A : \mathbb{R}_+ \to \mathbb{R}_{++}$ defined by $r_A(\cdot) = -u''(\cdot)/u'(\cdot)$. It is non-increasing ($r_A'(\cdot) \geq 0$) only if $u'''(\cdot) \geq -u''(\cdot)\,r_A(\cdot) > 0$.

$n$th dividend. This diminishes the agent's perceived "riskiness" of the $n$th security, inducing her to demand more of it and (in the face of fixed supply) raise its price in equilibrium.

A concrete example of this type of equilibrium price dynamics due to the asset-riskness effect is provided by Corollary 1.2. It assumes that the $n$th dividend and that of some other security, say the $n'$th, vary with the $m$th and the $k$th Brownian motions, respectively, while the former Brownian component is the only source of stochastic variations in the $n$th dividend ($\sigma_n = \sigma_{jm}\mathbf{e}_m$).[21] Moreover, these two Brownian motions do not affect other components of the terminal-period wealth ($\frac{\partial \rho(\mathcal{I}(T))}{\partial \beta_k(t)} = \frac{\partial \rho(\mathcal{I}(T))}{\partial \beta_m(t)} = 0$ and $\sigma_{n''k} = \sigma_{n''m} = 0$ for any $n'' \in \{1, \ldots, N\} \setminus \{n, n'\}$). The corresponding terminal-period wealth specification is a special case of (11)

$$\begin{aligned} W\left(\mathcal{I}\left(t\right), \mathbf{x}\right) &= \rho\left(\mathcal{I}\left(t\right), \mathbf{x}_{-(k,m)}\right) + \sum_{n'' \in \{1,\ldots,N\} \setminus \{n,n'\}} D_{n''}\left(\mathcal{I}\left(t\right), \mathbf{x}_{-(k,m)}\right) \\ &\quad + e^{\mu_{n'} T + \sigma_{n'k}\left(\beta_{n'k}(t) + \sqrt{T-t} x_k\right)} + e^{\mu_n T + \sigma_n\left(\beta_m(t) + \sqrt{T-t} x_m\right)} \\ &\equiv W_{-(k,m)}\left(\mathcal{I}\left(t\right), \mathbf{x}_{-(k,m)}\right) \\ &\quad + D_{n'}\left(\mathcal{I}\left(t\right), x_k\right) + D_n\left(\mathcal{I}\left(t\right), x_m\right) \end{aligned} \tag{13}$$

In this case, under DARA, the relative equilibrium price of the $n$th security is increasing (decreasing) in the realization $\beta_k(t)$ if $\sigma_{n'k} > 0$ ($\sigma_{n'k} < 0$). And this obtains even though the wealth effect on the relative price has the same sign as the wealth effect on the price of the bond, negative (positive) if $\sigma_{n'k} > 0$ ($\sigma_{n'k} < 0$).[22] Clearly, the monotonicity of $p_n(t)$ with respect to $\beta_k(t)$ is due to the fact that the asset-riskness effect of $\beta_k(t)$ on

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[21] As usual, $\mathbf{e}_m \in \mathbb{R}^K$ denotes the vector with 1 at its $m$th entry and zeroes elsewhere. Moreover, $\mathbf{x}_{-m} = (x_1, ..., x_{m-1}, x_{m+1}, ..., x_K)^\intercal \in \mathbb{R}^{K-1}$ and $\mathbf{x}_{-(k',m)} = (x_1, ..., x_{k'-1}, x_{k'+1}, ..., x_{m-1}, x_{m+1} ..., x_K)^\intercal \in \mathbb{R}^{K-2}$.

[22] Recall that the wealth effect of $\mathrm{d}\beta_k(t)$ operates in the same direction on all absolute prices. To establish that it pulls also all relative prices in this direction, it suffices to show that it drives the $n$th relative price in the direction in which it pushes the price of the bond. It is enough, therefore, that the expression in the brackets on the right-hand side of (10) be positive. Which follows immediately by risk aversion ($u_2''(\cdot) < 0$): (13) gives $W\left(\mathcal{I}(t), \mathbf{x} + \sqrt{T-t}\sigma_n\right) = W\left(\mathcal{I}(t), \mathbf{x}\right) + D_n\left(\mathcal{I}(t), x_m + \sqrt{T-t}\sigma_{jm}\right) - D_n\left(\mathcal{I}(t), x_m\right)$ where $D_n\left(\mathcal{I}(t), x_m + \sqrt{T-t}\sigma_{jm}\right) = e^{\mu_n T + \sigma_{jm}\left(\beta_m(t) + \sqrt{T-t}\left(y_m + \sqrt{T-t}\sigma_{jm}\right)\right)} = \left(e^{(T-t)\sigma_{jm}^2} - 1\right) D_n\left(\mathcal{I}(t), x_m\right)$.

$p_n(t)$ dominates the wealth effect.

Once we allow the $n$th dividend to depend upon the $k$th Brownian motion ($\sigma_{nk} \neq 0$), the mechanics of the asset-riskness effect become more complicated. Given a change $\mathrm{d}\beta_k(t)$, the new level of terminal-period wealth will be $W(\mathcal{I}(t), \mathbf{x}) + \mathrm{d}W(\mathcal{I}(t), \mathbf{x})$ while the new covariance of its marginal utility with the $n$th terminal dividend is given by

$$\begin{aligned} & \mathrm{Cov}_{\mathbf{x}}\left[u'\left(W(\mathcal{I}(t), \mathbf{x}) + \mathrm{d}W(\mathcal{I}(t), \mathbf{x})\right), e^{\mu_n T + \sigma_n^{\intercal}\left(\beta(t) + \mathrm{d}\beta_k(t)\mathbf{e}_k + \sqrt{T-t}\mathbf{x}\right)}\right] \\ = \; & e^{\sigma_{nk}\mathrm{d}\beta_k(t)}\mathrm{Cov}_{\mathbf{x}}\left[u'\left(W(\mathcal{I}(t), \mathbf{x}) + \mathrm{d}W(\mathcal{I}(t), \mathbf{x})\right), D_n(\mathcal{I}(t), \mathbf{x})\right] \end{aligned}$$

Obviously, what happens to the perceived "riskness" of the $n$th stock is determined now, not only by the covariance on the right-hand side of the above equation, but also by the term $e^{\sigma_{nk}\mathrm{d}\beta_k(t)}$.

Suppose, for instance, that $W(\mathcal{I}(t), \mathbf{x})$ and $D_n(\mathcal{I}(t), \mathbf{x})$ are again strictly comonotonic in $\mathbf{x}$. As before, $u'(W(\mathcal{I}(t), \mathbf{x}))$ is strictly countermonotonic in $\mathbf{x}$ and, thus, negatively correlated with $D_n(\mathcal{I}(t), \mathbf{x})$. Let also $\sigma_{nk}\mathrm{d}\beta_k(t) > 0$ so that $e^{\sigma_{nk}\mathrm{d}\beta_k(t)} > 1$. Even if, as in the preceding example, the change in terminal-period wealth renders its marginal utility less negatively correlated with the $n$th dividend, the increase in the dividend's drift might be sufficient to make their new covariance more negative overall. As opposed to the preceding example, the perceived "riskiness" of the $n$th stock would increase with $\beta_k(t)$, exerting a downward pressure on its equilibrium relative price.

The direction and importance of the asset-riskiness effect for the relative price dynamics depends also on the agent's utility function; namely, her risk-aversion. Consider, for instance, the following setting. The agent exhibits CARA and the $m$th Brownian motion affects both the $n$th and $n'$th terminal dividends. The former dividend is independent of any other Brownian component ($\sigma_n = \sigma_{nm}\mathbf{e}_m$). The latter varies also with but only with the $k$th Brownian motion ($\sigma_{n'} = \sigma_{n'm}\mathbf{e}_m + \sigma_{n'k}\mathbf{e}_k$), which, in turn, affects no other component of wealth ($\frac{\partial\rho(\mathcal{I}(T))}{\partial\beta_k(t)} = 0$ and $\sigma_{ik} = 0 \; \forall i \in \{1, \ldots, N\} \setminus \{n'\}$). The corresponding wealth specification is another subcase of (11):

$$\begin{aligned} W\left(\mathcal{I}\left(t\right),\mathbf{x}\right) &= \rho\left(\mathcal{I}\left(t\right),\mathbf{x}_{-k}\right)+\sum_{n''\in\{1,\ldots,N\}\setminus\{n,n'\}} D_{n''}\left(\mathcal{I}\left(t\right),\mathbf{x}_{-k}\right) \\ &\quad +e^{\mu_n T+\sigma_n\left(\beta_m(t)+\sqrt{T-t}x_m\right)} \\ &\quad +e^{\mu_{n'}T+\sigma_{n'm}\beta_m(t)+\sigma_{n'k}\beta_k(t)+\sqrt{T-t}(\sigma_{n'm}x_m+\sigma_{n'k}x_k)} \\ &\equiv W_{-k}\left(\mathcal{I}\left(t\right),\mathbf{x}_{-k}\right) \\ &\quad +D_n\left(\mathcal{I}\left(t\right),x_m\right)+D_{n'}\left(\mathcal{I}\left(t\right),\left(x_k,x_m\right)\right) \end{aligned} \tag{14}$$

In this setting, Corollary 2.1 dictates that, as long as $\sigma_{nm}\sigma_{n'm}>0$, a rise in $\beta_k\left(t\right)$ increases (decreases) the $n$th relative price if $\sigma_{n'k}<0$ ($\sigma_{n'k}>0$). To analyze this result in terms of the asset-riskness and wealth effects, we need to determine the direction of the latter. Which is easy to do if we restrict attention to the special case of (14) in which the $m$th Brownian motion affects no other component of the terminal-period wealth but the two dividends ($\frac{\partial\rho(\mathcal{I}(T))}{\partial\beta_m(t)}=0$ and $\sigma_{im}=0$ for any $i\in\{1,\ldots,N\}\setminus\{n,n'\}$).

The specification in question

$$\begin{aligned} & W\left(\mathcal{I}\left(t\right),\mathbf{x}\right) \\ = \; & W_{-(k,m)}\left(\mathcal{I}\left(t\right),\mathbf{x}_{-(k,m)}\right)+D_n\left(\mathcal{I}\left(t\right),x_m\right)+D_{n'}\left(\mathcal{I}\left(t\right),\left(x_k,x_m\right)\right) \end{aligned}$$

gives

$$\begin{aligned} & W\left(\mathcal{I}\left(t\right),\mathbf{x}+\sqrt{T-t}\sigma_n\right)-W_{-(k,m)}\left(\mathcal{I}\left(t\right),\mathbf{x}_{-(k,m)}\right) \\ = \; & \left(e^{(T-t)\sigma_{nm}^2}-1\right)D_n\left(\mathcal{I}\left(t\right),x_m\right)+\left(e^{(T-t)\sigma_{n'm}\sigma_{nm}}-1\right)D_{n'}\left(\mathcal{I}\left(t\right),\left(x_m,x_k\right)\right) \end{aligned}$$

so that, if $\sigma_{nm}\sigma_{n'm}>0$, we get $W\left(\mathcal{I}\left(t\right),\mathbf{x}+\sqrt{T-t}\sigma_n\right)>W\left(\mathcal{I}\left(t\right),\mathbf{x}\right)$. By (10), then, the wealth effect of $\mathrm{d}\beta_k\left(t\right)$ on the $n$th relative price operates in the same direction as it does on the price of the bond. Yet, $\sigma_{n'k}\frac{\partial W(\mathcal{I}(t),\mathbf{x})}{\partial\beta_k(t)}>0$ and (8) dictates that the wealth effect pushes the bond price in the direction of $\mathrm{d}\beta_k\left(t\right)$ if $\sigma_{n'k}<0$ ($\sigma_{n'k}>0$). Contrary to the DARA example, therefore, a change in $\beta_k\left(t\right)$ changes here the $n$th relative price in the direction of the wealth effect, irrespectively of the asset-riskness effect.

**The Combined Effect**

Recall (5). The dynamics of the $n$th relative price with respect to $\beta_k(t)$ are determined by two terms: the own-dividend effect, and the asset-riskiness effect relative to the price of the bond. As, however,

$$\frac{1}{p_n(t)}\frac{\partial p_n(t)}{\partial \beta_k(t)} = \frac{1}{P_n(t)}\frac{\partial P_n(t)}{\partial \beta_k(t)} - \frac{1}{P_0(t)}\frac{\partial P_0(t)}{\partial \beta_k(t)} \tag{15}$$

the dynamics are in fact given by the difference between the relative (percentage) changes in the absolute prices, $P_n(t)$ and $P_0(t)$; a complex enough relation, in general, to preclude its prediction using only economic intuition, mainly for three reasons. First, the wealth effects on the two absolute prices, by pushing them in the same direction, pull $p_n(t)$ in opposite directions. Second, the own-dividend effect on $P_n(t)$ pushes it always in the opposite direction than its wealth effect. Finally, as shown by the preceding examples, if $u(\cdot)$ exhibits NARA, the asset-riskiness effect may pull $p_n(t)$ in the opposite direction than the wealth effect.

Theorem 1 (in the next section) addresses these issues unequivocally for the dynamics of the typical relative price with respect to the current realization of the entire Brownian vector. It dictates that the inner product of the $n$th row of the dispersion matrix $\Sigma$ with the gradient vector of the $n$th relative price, $\nabla_{\beta(t)} p_n(t)$, is strictly positive as long as the $n$th dividend is stochastic, in the sense in which the uncertainty is captured in this model.[23]

The intuition behind this result is straightforward when the terminal dividend is correlated with only the $m$th Brownian motion and this relation is exclusive ($\sigma_n = \sigma_{nm}\mathbf{e}_m$, $\frac{\partial \rho(\mathcal{I}(T))}{\partial \beta_m(t)} = 0$, and $\sigma_{n'm} = 0\ \forall n' \in \{1, \ldots, N\} \setminus \{n\}$). The corresponding terminal-wealth specification is that in (11) for $M = 1$. In this setting, let $\beta_m(t)$ change by $\mathrm{d}\beta_m(t)$. For any realization

[23] For $\sigma_n = \mathbf{0}$, we get $p_n(t) = e^{\mu_n T}$. The relative price is constant, independent of any Brownian realization. Consider the typical Brownian dimension. Since $\sigma_{nk} = 0$, there is no own-dividend effect on $P_n(t)$. Since all other factor loadings of the $n$th terminal dividend are also zero, the dividend is independent of the subsequent path $\{\beta(\omega, \tau) : \tau \in (t, T]\}$ of the Brownian process and, consequently, of the terminal-period wealth. Clearly, a change in $\beta_k(t)$ induces no asset-riskiness effect on $P_n(t)$ while its wealth effects on $P_n(t)$ and $P_0(t)$ cancel each other out.

$x_m$, the terminal-period wealth changes now only through the $n$th dividend, whose new value is

$$\begin{aligned} D_n\left(\beta_m(t) + \mathrm{d}\beta_m(t), t, x_m\right) &= e^{\mu_n T + \sigma_{jm}\left(\beta_m(t) + \mathrm{d}\beta_m(t) + \sqrt{T-t}x_m\right)} \\ &= e^{\sigma_{nm}\mathrm{d}\beta_m(t)} D_n\left(\mathcal{I}(t), x_m\right) \end{aligned}$$

Since the agent is everywhere non-satiated ($u'(\cdot) > 0$) and any other component of her terminal-period wealth remains unaffected by $\mathrm{d}\beta_m(t)$, her preferences for the $n$th stock change in the direction of First-order Stochastic Dominance (FSD).

Suppose, specifically, that $\beta_m(t)$ increases (decreases). If $\sigma_{nm} > 0$, the new terminal dividend dominates (is dominated by) the old in the sense of FSD. The agent is now more (less) willing to hold the stock and, facing its fixed supply, pushes up its absolute price. By (8), in addition, the wealth effect on the price of the bond is negative (positive). Clearly, the relative price of the security increases (decreases). If $\sigma_{nm} < 0$, on the other hand, the new terminal dividend is dominated by (dominates) the old in terms of FSD whereas the wealth effect on $P_0(t)$ is positive (negative). In either case, therefore, $\sigma_{nm}\frac{\partial p_n(t)}{\partial \beta_m(t)} > 0$.[24]

In more complex settings, the theorem can be viewed as generalizing this argument to the relative price dynamics with respect to the current realization of entire Brownian vector. Its proof (see Appendix C) uses straightforward mathematical apparatus but is quite subtle in its reasoning, especially with respect to its last and most crucial step. It attests to the complexity of the equilibrium relation between the relative prices and the current realization of the underlying stochastic process.

[24] Put differently, when $\sigma_{nm} > 0$ ($\sigma_{nm} < 0$), going from the old to the new terminal dividend is in the opposite (same) direction as Proposition 1 in Gollier [18], the factor being $e^{\sigma_{nm}\mathrm{d}\beta_m(t)}$. For any risk-averse individual, $\mathrm{d}\beta_m(t)$ increases (reduces) the optimal demand and, consequently, the $n$th equilibrium relative price. Of course, Gollier studies probability distributions whose supports are closed intervals but this restriction is inconsequential in my context (see Lemma A.1 in Appendix A).

# 4 Dynamics of Relative Prices

Given the complexity of the dynamics in question, we cannot but restrict attention to situations in which there is sufficient structure for precise conclusions to be made. In what follows, my aim is to identify conditions on the economic primitives of the model that suffice for $p_n(t)$ to be monotone in $\beta_k(t)$. To this end, the building block of my analysis will be a result that holds universally across the space of economic primitives. The required conditions for it to apply are extremely mild, met by all utility functions generally of interest in financial economics.

**Theorem 1** *Let the nth terminal dividend be given by (4). Suppose also that, given $\beta_{-k}(t) \in \mathbb{R}^{K-1}$ and viewing $u'(W(\mathcal{I}(t), \mathbf{x})) D_n(\mathcal{I}(t), \mathbf{x})$ as a function $\mathbb{R}^{K+1} \mapsto \mathbb{R}_{++}$ of $(\beta_k(t), \mathbf{x})$, Lemma A.1 in Appendix A applies. Then,*

$$\sum_{k=1}^{K} \sigma_{nk} \frac{\partial p_n(t)}{\partial \beta_k(t)} \geq 0 \qquad \textit{with equality only if } \sigma_n = \mathbf{0}$$

This claim refers to the typical row of the Jacobian matrix of relative prices

$$J_p(t) = \left[\frac{\partial p_n(t)}{\partial \beta_k(t)}\right]_{(n,k)\in\{1,\ldots,N\}\times\{1,\ldots,K\}}$$

not to its typical element. Yet, it has immediate implications for the dynamics of the typical relative price when the associated dividend varies with the terminal realization of only one Brownian motion ($\sigma_n = \sigma_{nm}\mathbf{e}_m$ for some $m \in \{1, ..., K\}$). Specifically, it follows immediately from the theorem that, given

$$D_n(\mathcal{I}(t), \mathbf{x}) = e^{\mu_n T + \sigma_{nm}\left(\beta_m(t) + \sqrt{T-t}x_m\right)} \tag{16}$$

$p_n(t)$ will be monotone in $\beta_m(t)$ so that the observed path of the former identifies uniquely the path $\{\beta_m(\tau) : \tau \in (t, T]\}$ in which the associated uncertainty gets resolved. More precisely, we have $\sigma_{nm} \frac{\partial p_n(t)}{\partial \beta_m(t)} > 0$.

In this case, the combination of the three potentially contradicting effects highlighted in the preceding section is identified unequivocally by the theorem. To illustrate, let the agent exhibit DARA and her terminal wealth be increasing in the current realization of the $m$th Brownian motion, $\frac{\partial W(\mathcal{I}(T))}{\partial \beta_m(t)} > 0$ (which would be the case, for example, if $\sigma_{nm} > 0$ and $\frac{\partial \rho(\mathcal{I}(T))}{\partial \beta_m(t)}, \sigma_{n',m} \geq 0$ $\forall n' \in \{1, \ldots, N\} \setminus \{n\}$). If the $n$th terminal dividend is positively correlated with the $m$th Brownian component ($\sigma_{nm} > 0$), an increase in $\beta_m(t)$ raises its $\mathcal{F}_t$-conditional expectation, pushing its price $P_n(t)$ upwards through the own-dividend effect. It increases, though, also the agent's terminal wealth, exerting negative wealth effects on both $P_0(t)$ and $P_n(t)$. And, as pointed out in the previous section, the asset-riskiness effect on $P_n(t)$ may go in either direction. Nevertheless, the combined effect on the latter price is such that, even though the price of the bond necessarily falls, that of the stock either increases or decreases by less in percentage terms.

More generally, the theorem describes completely the price dynamics of the economy when there is a single source of uncertainty and one tree ($N = K = 1$), a model representing stocks and bonds as broad asset classes. It applies also to the dynamics of every risky security in the model, with respect to the associated risk source, when $\Sigma$ is diagonal ($N = K$ and $\Sigma = [\sigma_{11}\mathbf{e}_1, ..., \sigma_{KK}\mathbf{e}_K]$), in which case $\sigma_{nn}\frac{\partial p_n(t)}{\partial \beta_n(t)} > 0$ $\forall n = 1, \ldots, K$.

## 4.1 Contagion

Having identified the relation between the relative price and the associated Brownian motion when the dividend depends on only one Brownian component, the obvious next step is to examine it with respect to $\beta_k(t)$, for some $k \neq m$. In what follows, I will present some results which, in conjunction with Theorem 1, describe the comparative statics of the corresponding economy. Their common theme is that, apart from quite special cases, the relative price $p_n(t)$ varies with $\beta_k(t)$ when $\sigma_{nk} = 0$. How it does depends on (i) the way in which the terminal wealth depends upon the terminal realization of the Brownian process, and (ii) the functional form (the risk-attitude in particular) of the agent's utility function.

Given that $\sigma_{nk} = 0$, changes in $\beta_k(t)$ produce no own-dividend effect on the absolute price $P_n(t)$, only wealth and asset-riskiness effects. The derivative of interest then becomes

$$\frac{\partial p_n(t)}{\partial \beta_k(t)} = \left( \frac{e^{\mu_n T + \sigma_n^{\intercal} \beta(t)}}{P_0(t)^2 \sqrt{(T-t)(2\pi)^{2K}}} \right) \tag{17}$$
$$\mathrm{Cov}_{y_k} \left[ y_k, \mathbb{E}_{(\mathbf{x}, \mathbf{y}_{-k})} \left[ \begin{array}{c} u'\left(W\left(\mathcal{I}(t), \mathbf{y} + \sqrt{T-t}\sigma_n\right)\right) u'\left(W\left(\mathcal{I}(t), \mathbf{x}\right)\right) \\ -u'\left(W\left(\mathcal{I}(t), \mathbf{x} + \sqrt{T-t}\sigma_n\right)\right) u'\left(W\left(\mathcal{I}(t), \mathbf{y}\right)\right) \end{array} \right] \right]$$

**Contagion under DARA**

The dynamics of the $n$th relative price with respect to changes in the Brownian realization $\beta_k(t)$ when $\sigma_{nk} = 0$ are particularly rich. Enough so, in fact, to render contagion in this representative agent economy a rather generic phenomenon regarding her utility function. For, as I show below, under any DARA utility, the relative price varies (monotonically) with the current realization of the $k$th Brownian motion even when the dispersion matrix $\Sigma$ is diagonal and the terminal-period endowment is deterministic.

To demonstrate the prevalence of contagion due to market-clearing, I will progressively stack the cards against contagion. Let us begin, therefore, by assuming that the $k$th Brownian motion affects the agent's wealth only through dividends and, in particular, ones that are not correlated with any of the Brownian motions that affect the payoff of the $n$th stock. To state this formally, recall that the condition $\sigma_{nk} = 0$ can be equivalently written as $k \notin K_n$, for the index set of those Brownian components that are correlated with the $n$th terminal dividend. Let also

$$N_k = \left\{ n' \in \{1, \ldots, N\} : \sigma_{n'k} \neq 0 \right\}$$

denote the index set of those stocks whose terminal dividends do vary with the $k$th Brownian motion. We require then $\frac{\partial \rho(\mathcal{I}(T))}{\partial \beta_k(T)} = 0$ and $N_m \cap N_k = \varnothing$

$\forall m \in K_n$, the corresponding wealth specification being

$$\begin{aligned} W\left(\mathcal{I}\left(t\right),\mathbf{x}\right) &= \rho\left(\mathcal{I}\left(t\right),\mathbf{x}_{-k}\right) \\ &\quad + \sum_{n'\in N_k} D_{n'}\left(\mathcal{I}\left(t\right),\mathbf{x}_{-M}\right) + \sum_{n''\in N_m} D_{n''}\left(\mathcal{I}\left(t\right),\mathbf{x}_{-k}\right) \end{aligned} \tag{18}$$

As it turns out, under some additional restrictions, the $n$th relative price varies (monotonically) with the $k$th Brownian motion under DARA.

**Proposition 1** *Let the following conditions apply.*

*(i)* $u(\cdot)$ *exhibits DARA while the* $n$*th terminal dividend and the terminal wealth are given by (4) and (18), respectively.*

*(ii)* $\sigma_{n'm} = \sigma_{nm}\ \forall (n', m) \in N_m \times K_n$.

*(iii)* $\sigma_{n'k}\sigma_{n''k} > 0\ \forall n', n'' \in N_k$. *Then*

$$\sigma_{n'k}\frac{\partial p_n(t)}{\partial \beta_k(t)} > 0 \quad n' \in N_k$$

The covariance matrix $\Sigma_1$ depicts a situation within the operational realm of this claim. It refers to an economy where the first risky security is an exclusive "bet" on the first Brownian component, a risk factor which does not affect any other asset. The result applies on the relative price of this stock and for $k \geq 2$, as long as the terminal-period endowment is independent of the first and the $k$th Brownian components and $\sigma_{2k}\sigma_{3k} > 0$. The inequality is due to condition (iii) while condition (ii) is redundant since the index set $N_1$ contains only the first security. In this case, we have $\sigma_{2k}\frac{\partial p_1(t)}{\partial \beta_k(t)} > 0$.

$$\Sigma_1 = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & \sigma_{32} & \sigma_{33} \end{pmatrix} \qquad \Sigma_2 = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}$$

If the terminal-period endowment is independent of either of the last two Brownian components, the requirements in the preceding paragraph and, thus, its last relation may hold for $k = 2, 3$. As, in addition, $\sigma_{11}\frac{\partial p_1(t)}{\partial \beta_1(t)} > 0$

(Theorem 1), we can now sign the entire first row of the Jacobian matrix of relative prices. If, moreover, the terminal-period endowment is deterministic, we can sign also the derivatives of the second and third relative prices with respect to the first Brownian motion. For these cases, condition (iii) is redundant ($N_1$ is a singleton) while condition (ii) requires that $\sigma_{2k} = \sigma_{3k}$ for $k = 2, 3$. We ought to have then $\sigma_{11} \frac{\partial p_n(t)}{\partial \beta_1(t)} > 0$ for $n = 2, 3$.

The application of Proposition 1 on the first relative price of $\Sigma_1$ brings us forward in our quest to stack the cards of our model as much as possible against cross-correlations. For it indicates that cross-correlations obtain even when the payoff of $n$th risky security is correlated with only one Brownian component ($K_n = \{m\}$).

**Corollary 1.1** *Let the following conditions apply.*

*(i)* $u(\cdot)$ *exhibits DARA while the* $n$*th terminal dividend and the terminal wealth are given by (16) and (18), respectively.*

*(ii)* $\sigma_{\widetilde{n}m} = \sigma_{nm}\ \forall \widetilde{n} \in N_m$.

*(iii)* $\sigma_{n'k}\sigma_{n''k} > 0\ \forall n', n'' \in N_k$. *Then*

$$\sigma_{n'k} \frac{\partial p_n(t)}{\partial \beta_k(t)} > 0 \quad n' \in N_k$$

Before proceeding further, I should point out that this corollary assumes the terminal-wealth specification in (18), which now reads

$$W(\mathcal{I}(t), \mathbf{x}) = \rho(\mathcal{I}(t), \mathbf{x}_{-k}) + \sum_{n' \in N_k} D_{n'}(\mathcal{I}(t), \mathbf{x}_{-m}) + \sum_{n'' \in N_m} D_{n''}(\mathcal{I}(t), \mathbf{x}_{-k})$$

mainly for expositional ease in the presentation of its proof (see Appendix C). The result does apply, for instance, also when

$$\begin{aligned} W(\mathcal{I}(t), \mathbf{x}) &= \rho_1\left(\beta_{-m}(t) + \sqrt{T-t}\mathbf{x}_{-m}\right) + \rho_2\left(\beta_m(t) + \sqrt{T-t}x_m\right) \\ &+ \sum_{n' \in \{1,\dots,N\} \setminus \{n\}} D_{n'}(\mathcal{I}(t), \mathbf{x}_{-m}) + D_n(\mathcal{I}(t), x_m) \quad (19) \end{aligned}$$

for some continuous functions $\rho_1 : \mathbb{R}^{K-1} \mapsto \mathbb{R}_+$ and $\rho_2 : \mathbb{R} \mapsto \mathbb{R}_+$ such that

$$\rho\left(\beta\left(T\right), T\right) = \rho_1\left(\beta_{-m}\left(T\right), T\right) + \rho_2\left(\beta_m\left(T\right), T\right)$$

as long as each of the derivatives $\frac{\partial W(\mathcal{I}(t), \mathbf{x}_{-m})}{\partial x_k}$ and $\frac{\partial W(\mathcal{I}(t), x_m)}{\partial x_m}$ maintains a given sign on $\mathbb{R}$.

Specifically, let $\lambda_k \frac{\partial W(\mathcal{I}(t), \mathbf{x}_{-m})}{\partial x_k}, \lambda_m \frac{\partial W(\mathcal{I}(t), x_m)}{\partial x_m} > 0$ for some $\lambda_k, \lambda_m \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^K$. As we already know, the wealth effect of the realization $\beta_k\left(t\right)$ pushes both equilibrium prices $P_0\left(t\right)$ and $P_n\left(t\right)$ in the direction in which it moves the terminal wealth. Given the separability in (19), this direction is given by the derivative $\frac{\partial W(\beta_{-m}(T), T)}{\partial \beta_k(t)}$ (i.e., by the sign of $\lambda_k$). By contrast, the specification in (19) being a special case of that in (11), the asset-riskiness effect of $\beta_k\left(t\right)$ on the relative price $p_n\left(t\right)$ is given by (12) as

$$\mathbb{E}_{x_k}\left[\mathrm{Cov}_{\mathbf{x}_{-k}}\left[u''\left(W\left(\mathcal{I}\left(t\right), \mathbf{x}\right)\right), D_n\left(\mathcal{I}\left(t\right), x_m\right)\right] \frac{\partial W\left(\mathcal{I}\left(t\right), \mathbf{x}_{-m}\right)}{\partial \beta_k\left(t\right)}\right]$$

The combined effect on the $n$th equilibrium relative price is to change it monotonically. It is straightforward to reproduce the proof of Corollary 1.1 in this setting and verify that $\sigma_{nm}\lambda_m\lambda_k \frac{\partial p_n(t)}{\partial \beta_k(t)} > 0$.

To continue strengthening the model against contagion, we may revisit the terminal wealth specification in (13), the subcase of the one in (19) which restricts the $k$th and $m$th Brownian components to be correlated with no other terminal-wealth components but the $n'$th and $n$th terminal dividends, respectively, for some $n' \neq n$. Under such a requirement, both sets $K_n$ and $N_k$ are singletons so that conditions (ii)-(iii) of Corollary 1.1 become redundant, allowing it to be stated as follows.

**Corollary 1.2** *Suppose that $u\left(\cdot\right)$ exhibits DARA while the $n$th terminal dividend and the terminal wealth are given by (16) and (13), respectively. Then*

$$\sigma_{n'k} \frac{\partial p_n\left(t\right)}{\partial \beta_k\left(t\right)} > 0$$

This result applies even under the most restrictive $\Sigma$-form against cross-correlation in relative prices. Namely, the case of a diagonal matrix of factor loadings, such as $\Sigma_2$, where the claim is valid for any security $n$ and any Brownian motion $k \neq n$ as long as the terminal-period endowment is uncorrelated with either of the $n$th and $k$th Brownian components. If, in particular, the terminal-period endowment is deterministic, the corollary along with Theorem 1 allow us to sign the entire Jacobian matrix of the relative price process. Under a diagonal $\Sigma$ of general dimensions and a deterministic terminal-period endowment, the terminal-wealth specification is given by

$$W\left(\mathcal{I}\left(t\right),\mathbf{x}\right)=\rho\left(T\right)+\sum_{n=1}^{N}D_n\left(\mathcal{I}\left(t\right),x_n\right) \tag{20}$$

The entries of $J_p\left(t\right)$ are such that $\sigma_{kk}\frac{\partial p_n(t)}{\partial \beta_k(t)}>0$ for $k=1,\ldots,K$.

### Contagion under CARA

Cross-correlations will generally be nonzero even when the representative agent exhibits CARA. And, even in this case, there are settings of economic primitives where the cross-derivative of the relative price maintains everywhere the same sign, so that $p_n\left(t\right)$ remains monotone in $\beta_k\left(t\right)$ when $k \notin K_n$. To demonstrate the prevalence of contagion due to market-clearing under CARA, I will again progressively stack the cards against contagion, starting now with the hypothesis that the $k$th Brownian motion affects the agent's wealth only through dividends ($\frac{\partial \rho(\mathcal{I}(T))}{\partial \beta_k(T)}=0$).

Under the corresponding terminal-wealth specification

$$\begin{aligned}W\left(\mathcal{I}\left(t\right),\mathbf{x}\right) &= \rho\left(\mathcal{I}\left(t\right),\mathbf{x}_{-k}\right)\\ &\quad+\sum_{n'\in N_k}D_{n'}\left(\mathcal{I}\left(t\right),\mathbf{x}\right)+\sum_{n''\notin N_k}D_{n''}\left(\mathcal{I}\left(t\right),\mathbf{x}_{-k}\right)\end{aligned} \tag{21}$$

which actually embeds the one in (18), we have the following result.

**Proposition 2** *Suppose that the following conditions apply.*

*(i)* $u\left(\cdot\right)$ *exhibits CARA* ($u\left(c\right)=\gamma e^{\alpha c} \quad \gamma,\alpha<0$) *while the* $n$*th terminal dividend and the terminal wealth are given by (4) and (21), respec-*

*tively.*

(ii) $\forall (n', m) \in N_k \times K_n$, $\exists \lambda_{n'} \in \mathbb{R}^*$: $\sigma_{nm} = \lambda_{n'} \sigma_{n'm}$

(iii) $\lambda_{n'} \sigma_{n'k} \lambda_{n''} \sigma_{n''k} > 0 \ \forall n', n'' \in \cup_{m \in K_n} (N_m \cap N_k)$. *Then*

$$\sigma_{n'm} \sigma_{nm} \sigma_{n'k} \frac{\partial p_n(t)}{\partial \beta_k(t)} < 0 \quad \forall n' \in N_m \cap N_k$$

As shown in Appendix C, when the $n$th dividend is correlated with only one Brownian motion, condition (ii) above becomes redundant and the statement simplifies as follows.

**Corollary 2.1** *Let the following apply.*

(i) $u(\cdot)$ *exhibits CARA while the nth terminal dividend and the terminal wealth are given by (16) and (21), respectively.*

(ii) $\prod_{n' \in N_m \cap N_k} \sigma_{n'm} \sigma_{n'k} > 0$. *Then*

$$\sigma_{nm} \sigma_{n'm} \sigma_{n'k} \frac{\partial p_n(t)}{\partial \beta_k(t)} < 0 \quad \forall n' \in N_m \cap N_k$$

To illustrate the workings of these claims, consider the dispersion matrix $\Sigma_3$, a generalization of $\Sigma_1$ depicting an economy where the first Brownian component represents macroeconomic uncertainty - it affects all risky assets (albeit with possibly different degrees of sensitivity) - while the first stock is a "bet," exclusively on this risk factor.

$$\Sigma_3 = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \qquad \Sigma_4 = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & 0 & \sigma_{33} \end{pmatrix}$$

Using the corollary, we can determine the dynamics of the relative price of the macroeconomic "bet" with respect to changes in the realization of one of the non-macroeconomic risk-factors ($k \geq 2$), as long as the agent's terminal-period endowment does not depend upon it. Condition (ii) of the corollary reads here $\sigma_{21}\sigma_{2k}\sigma_{31}\sigma_{3k} > 0$. In this case, we have $\sigma_{11}\sigma_{n'1}\sigma_{n'k}\frac{\partial p_1(t)}{\partial \beta_k(t)} < 0$ with $n' \in \{2, 3\}$. As in addition $\sigma_{11}\frac{\partial p_1(t)}{\partial \beta_1(t)} > 0$ by Theorem 1, we can actually sign the entire first row of the Jacobian matrix of the relative price process.

To deploy Proposition 2 as well, we may assume that $\sigma_{32} = 0$ in this example and sign also the derivative $\frac{\partial p_3(t)}{\partial \beta_2(t)}$. Now, $K_3 = \{1, 3\}$ and $N_2 = \{2\}$ so that condition (ii) of the proposition requires that $\sigma_{31}/\sigma_{21} = \sigma_{33}/\sigma_{23}$ while condition (iii) is redundant ($\cup_{m \in K_3} (N_m \cap N_2) = N_2$ since the latter set is a singleton).[25] Here, as long as the second Brownian motion is not correlated with the terminal endowment, it must be $\sigma_{2m}\sigma_{3m}\sigma_{22}\frac{\partial p_3(t)}{\partial \beta_2(t)} < 0$ with $m \in \{1, 3\}$.

To constrain the economic setup against cross-correlations more, suppose that also the factor loading $\sigma_{23}$ is zero in the preceding example, as depicted by the covariance matrix $\Sigma_4$. Now, for $k \in \{2, 3\}$, the $k$th Brownian component affects only one terminal dividend, the unique payoff that is correlated with both the $k$th and the first Brownian motion. In general, we may require that, for $k \neq m$, the $k$th Brownian motion affects no terminal-wealth element but, say, the $n'$th terminal dividend ($\frac{\partial \rho(\mathcal{I}(T))}{\partial \beta_k(T)} = 0$ and $N_k = \{n'\}$ for some $n' \neq n$). This dividend, moreover, is correlated only with the $k$th and $m$th Brownian motions ($\sigma_{n'} = \sigma_{n'm}\mathbf{e}_m + \sigma_{n'k}\mathbf{e}_k$).

The specification in question is given by (14), which is obviously embedded in (21). In this case, condition (ii) of the preceding corollary becomes redundant ($N_m \cap N_k$ is a singleton) and, as shown in Appendix C, the claim can be stated as follows.

**Corollary 2.2** *Suppose that $u(\cdot)$ exhibits CARA while the $n$th terminal dividend and the terminal wealth are given by (16) and (14), respectively. Then,*

$$\sigma_{nm}\sigma_{n'm}\sigma_{n'k}\frac{\partial p_n(t)}{\partial \beta_k(t)} < 0$$

In the example $\Sigma_4$, this applies for the relative price of the first stock with respect to $k \geq 2$ as long as the $k$th Brownian motion is not correlated with the terminal-period endowment process. In this case, we have

[25] Taking $m \in K_3 = \{1, 2\}$, we have $N_1 = \{1, 2, 3\}$ and $N_3 = \{2, 3\}$ so that $N_m \cap N_2 = \{2\}$. Condition (ii), therefore, requires that $\sigma_{31} = \lambda_2\sigma_{21}$ and $\sigma_{33} = \lambda_2\sigma_{23}$ for some $\lambda_2 \neq 0$. For the redundancy of condition (iii) when the set $\cup_{m \in K_n} (N_m \cap N_k)$ is a singleton, see the proof of the proposition in Appendix C - in particular, the concluding part which establishes Corollary 2.2. Notice also that, given $\sigma_{32} = 0$, condition (ii) of Corollary 2 reduces to $\sigma_{21}\sigma_{22} > 0$ for $k = 2$.

$\sigma_{11}\sigma_{k1}\sigma_{kk}\frac{\partial p_1(t)}{\partial \beta_k(t)} < 0$ for $k = 2, 3$.

**No Cross-correlations under CARA: a very special case**

The preceding results might seem puzzling at fist glance for they contradict a rather commonly held view: under CARA, changes in wealth that are independent of an asset's payoff should not matter for its equilibrium relative price. An assertion that stems probably from an unwarranted generalization of the applicability of the following fact. As is well known, under CARA, changes in wealth that do not affect the risk premium of an asset should leave its relative price unchanged. If the change in $\beta_k(t)$ results in such a wealth change, therefore, and given the absence of the own-dividend effect when $\sigma_{nk} = 0$, the asset-riskiness effect on the absolute price of the $n$th stock should exactly cancel out the wealth effect on its relative price.

A sufficient condition for this to happen is that the $k$th and $m$th Brownian components affect the agent's terminal wealth through independent channels. This obtains under either of two terminal-wealth specifications. In the first, the $k$th Brownian motion affects the agent's terminal wealth in an exclusive way. Specifically, it may be correlated with only one of the remaining $N-1$ terminal dividends with this dividend not correlated with any other Brownian component. The $k$th Brownian component may also affect the terminal-period endowment process but through an element that is uncorrelated with any of the other Brownian motions. Formally, let $\sigma_{n'} = \sigma_{n'k}\mathbf{e}_k$ and $N_k = \{n'\}$ for some $n' \neq n$ and suppose also that

$$\rho\left(\beta\left(T\right), T\right) = \rho_1\left(\beta_{-k}\left(T\right), T\right) + \rho_2\left(\beta_k\left(T\right), T\right)$$

for some continuous functions $\rho_1 : \mathbb{R}^{K-1} \mapsto \mathbb{R}_+$ and $\rho_2 : \mathbb{R} \mapsto \mathbb{R}_+$. The terminal wealth can now be expressed as

$$\begin{aligned} W\left(\mathcal{I}\left(t\right), \mathbf{x}\right) \quad &= \quad \rho_1\left(\left(\beta_{-k}\left(t\right) + \sqrt{T-t}\mathbf{x}_{-k}\right)\right) + \rho_2\left(\beta_k\left(t\right) + \sqrt{T-t}x_k\right) \\ &\quad + \sum_{n'' \neq n, n'} D_{n''}\left(\mathcal{I}\left(t\right), \mathbf{x}\right) \\ &\quad + D_n\left(\mathcal{I}\left(t\right), x_m\right) + D_{n'}\left(\mathcal{I}\left(t\right), x_k\right) \qquad (22) \end{aligned}$$

of which the formulations in (13) and (14) are subcases.

The second specification is the one in (19) in which the $m$th Brownian component affects the terminal wealth separately from the remaining $K-1$ Brownian motions. It does so, moreover, via exclusive relations with at most two terminal-wealth components: through the $n$th dividend and, possibly, through some component of the terminal-period endowment process.

When the agent exhibits CARA, under either of these specifications, changes in the $k$th component of the Brownian process leave the $n$th relative equilibrium price unaffected.[26]

**Proposition 3** *Suppose that $u(\cdot)$ exhibits CARA while the terminal wealth is specified as in (19) or (22). Then,*

$$\frac{\partial p_n(t)}{\partial \beta_k(t)} = 0$$

An important special case of the specifications in (19) or (22) obtains when the dispersion matrix $\Sigma$ is diagonal and the terminal-period endowment process is separable along the $K$ dimensions of the Brownian vector. Formally, the latter condition is that

$$\rho(\beta(T), T) = \sum_{i=1}^{K} \rho_i \left( \beta_i(t) + \sqrt{T-t} x_i \right)$$

for some continuous functions $\rho_i : \mathbb{R} \to \mathbb{R}_+$ and the corresponding terminal wealth specification is

$$\begin{aligned} W(\mathcal{I}(t), \mathbf{x}) &= \sum_{i=1}^{K} \rho_i \left( \beta_i(t) + \sqrt{T-t} x_i \right) \\ &\quad + \sum_{i=1}^{K} D_i(\mathcal{I}(t), x_i) \end{aligned} \tag{23}$$

The proposition requires now that $\frac{\partial p_n(t)}{\partial \beta_k(t)} = 0 \ \forall k \in \{1, \ldots, K\} \setminus \{n\}$. In

[26] It can be shown explicitly actually that, under either of the three terminal wealth specifications (22)-(23), $\beta_k(t)$ is not a functional argument of $p_n(t)$. See equations (30)-(32) in Appendix C.

view of Theorem 1, the Jacobian matrix of relative prices is diagonal with all diagonal elements being nonzero. It is nonsingular, therefore, and the securities market is dynamically complete.[27]

Proposition 3 appears to support the premise that, under CARA, changes in wealth that are independent of an asset's payoff should not matter for its relative price. Yet, the fact that $p_n(t)$ does not respond to changes in $\beta_k(t)$ is not only due to $\sigma_{nk} = 0$ and CARA. It depends also, and fundamentally so, upon the separability of the channels through which the $k$th and $m$th Brownian motions operate in (19) and (22). For we know from Proposition 2 and its subsequent corollaries that, as soon as the two Brownian components are allowed to influence the agent's wealth through a common element, the relative price will no longer be unresponsive to changes in $\beta_k(t)$, even though the CARA and $\sigma_{nk} = 0$ assumptions are maintained.

To see what is so special about the underlying separability in the specifications (19) and (22), it is instructive to consider equation (27) in Appendix C, which gives the rates of change of the equilibrium absolute prices of the $n$th risky security and the bond with respect to the current realization of the $k$th Brownian motion. For $\sigma_{nk} = 0$, since $\mathbb{E}[x_k] = 0$, it reads

$$\begin{aligned}\frac{\partial P_n(t)}{\partial \beta_k(t)} &= \frac{1}{T-t}\mathbb{E}_{\mathbf{x}}\left[\sqrt{T-t}x_k u'(W(\mathcal{I}(t),\mathbf{x}))D_n(\mathcal{I}(t),\mathbf{x})\right] \\ &= \frac{1}{T-t}\mathrm{Cov}_{\mathbf{x}}\left[\sqrt{T-t}x_k, u'(W(\mathcal{I}(t),\mathbf{x}))D_n(\mathcal{I}(t),\mathbf{x})\right] \\ \frac{\partial P_0(t)}{\partial \beta_k(t)} &= \frac{1}{T-t}\mathrm{Cov}_{\mathbf{x}}\left[\sqrt{T-t}x_k, u'(W(\mathcal{I}(t),\mathbf{x}))\right]\end{aligned}$$

Either equation is in terms of the $\mathcal{F}_t$-conditional covariance between the marginal utility of terminal wealth (and, thus, consumption) that is derived from holding an extra unit of the security and the Brownian increment $\beta_k(T) - \beta_k(t)$. It is trivial to check that, when the agent's utility exhibits CARA and her terminal wealth is given by either of (19) and (22), $\frac{\frac{\partial P_n(t)}{\partial \beta_k(t)}}{P_n(t)} = \frac{\frac{\partial P_0(t)}{\partial \beta_k(t)}}{P_0(t)}$. But then condition (15) precludes any changes in the relative price

[27]By Theorem 2, dynamic completeness requires in turn that the factor loadings matrix $\Sigma$ is invertible. Indeed, under the specification in (23), it is necessarily diagonal.

of the security.

In this case, a change in the realization $\beta_k(t)$ induces a percentage change in the covariance of the marginal utility of terminal wealth with the $n$th terminal dividend which is exactly equal to the percentage change it induces in the price of the bond. As a consequence, the covariance in question remains unchanged when measured in units of the bond, which means in turn that the second term on the right-hand side of (5) remains unaltered. And so does the relative price itself given that the expected terminal dividend does not vary with $\beta_k(t)$.

Most probably, the erroneously crude intuition behind the "zero cross-correlations under CARA" premise stems from the multitude of examples in the financial economics literature that take the agent's wealth to be linearly-dependent upon asset payoffs. Although rendering discrete-time models analytically tractable and elegant, the linearity assumption obscures our grasp of the interaction between the asset-riskiness and wealth effects on relative equilibrium prices. For it forces this interaction to amount to nothing. And this is true irrespectively of the correlations between the various other elements of the agent's wealth.

To illustrate, suppose that $W(T)$ is linear on the $k$th Brownian component: $\frac{\partial W(\mathcal{I}(t),\mathbf{x})}{\partial \beta_k(t)} = \lambda_k$ for some $\lambda_k \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^K$. From (12), the asset-riskiness effect on the relative equilibrium price is now

$$
\begin{aligned}
& \frac{\lambda_k}{P_0(t)} \mathrm{Cov}_{\mathbf{x}} \left[ u''\left(W\left(\mathcal{I}(t), \mathbf{x}\right)\right), D_n\left(\mathcal{I}(t), \mathbf{x}\right) \right] \\
= \; & \frac{\alpha \lambda_k}{P_0(t)} \mathrm{Cov}_{\mathbf{x}} \left[ u'\left(W\left(\mathcal{I}(t), \mathbf{x}\right)\right), D_n\left(\mathcal{I}(t), \mathbf{x}\right) \right] \\
= \; & \alpha \lambda_k \left( p_n(t) - \mathbb{E}_{\mathbf{x}} \left[ D_n\left(\mathcal{I}(t), \mathbf{x}\right) \right] \right)
\end{aligned}
$$

the first equality above following from CARA. But this is exactly the opposite of the wealth effect which is given by

$$\begin{aligned}
& \frac{1}{P_0(t)}\left(\mathbb{E}_{\mathbf{x}}\left[D_n\left(\mathcal{I}(t), \mathbf{x}\right)\right]-p_n(t)\right) \frac{\partial P_0(t)}{\partial \beta_k(t)} \\
= \quad & \frac{1}{P_0(t)}\left(\mathbb{E}_{\mathbf{x}}\left[D_n\left(\mathcal{I}(t), \mathbf{x}\right)\right]-p_n(t)\right) \mathbb{E}_{\mathbf{x}}\left[u''\left(W\left(\mathcal{I}(t), \mathbf{x}\right)\right) \frac{\partial W\left(\mathcal{I}(t), \mathbf{x}\right)}{\partial \beta_k(t)}\right] \\
= \quad & \alpha \lambda_k\left(\mathbb{E}_{\mathbf{x}}\left[D_n\left(\mathcal{I}(t), \mathbf{x}\right)\right]-p_n(t)\right)
\end{aligned}$$

Of course, given that the $\mathcal{F}_t$-conditional future realizations $\beta_k(T) - \beta_k(t)$ are normally-distributed here, the linearity assumption requires unlimited liability, an unrealistically strong condition (as it implies that the agent may lose more than everything with positive probability). This is a well-known drawback. To make matters worse, the assumption is restrictive also in a theoretical sense. When the representative agent exhibits CARA, it conditions the asset-riskiness and wealth effects on the relative equilibrium price to cancel each other out.

## 4.2 General Dynamics

To complete the investigation on the dynamics of the relative price process, it remains to consider the case $k \in K_n$ ($\sigma_{nk} \neq 0$). Evidently from our analysis thus far, when the terminal dividend is correlated with the Brownian dimension of interest, there is really little hope of pinpointing settings in which its relative price is monotone in the Brownian realizations. Nevertheless, I conclude by presenting a situation in which the correlation between the relative price of the security and the underlying Brownian motion maintains a constant sign throughout the stochastic domain.

**Claim 3.1** *Let the following conditions apply.*

*(i)* $u(\cdot)$ *exhibits CRRA* ($u(c) = \gamma c^{\alpha}$ $\alpha, \gamma < 0$ *or* $u(c) = \ln c$) *while the* $n$*th terminal dividend is given by (4).*

*(ii)* $\exists \lambda_n \in \mathbb{R}_{++}$*:* $W\left(\mathcal{I}(t), \mathbf{x} + \sqrt{T-t}\sigma_n\right) = \lambda_n W\left(\mathcal{I}(t), \mathbf{x}\right)$ $\forall \mathbf{x} \in \mathbb{R}^K$*. Then, setting* $\alpha = 0$ *in the logarithmic case,*

$$\frac{\partial p_n(t)}{\partial \beta_k(t)} = \sigma_{nk} \lambda_n^{\alpha-1} e^{\mu_n T + \sigma_n^{\intercal}\left(\beta - \frac{(T-t)\sigma_n}{2}\right)} \quad \forall k = 1, \ldots, K$$

Admittedly, the setting under which this result applies is quite specific. Yet, it is also instructive for it allows the recovery of the entire Jacobian matrix of relative prices, its $n$th row being (in column form)

$$\mathbf{j}_{p,n}(t) = \lambda_n^{\alpha-1} e^{\mu_n T + \sigma_n^\intercal \left(\beta(t) - \frac{(T-t)\sigma_n}{2}\right)} \sigma_n$$

An example of the relevant setting can be constructed by considering an economy in which the terminal-period endowment is deterministic while the factor loadings are such that $(\sigma_{n'} - \sigma_n)^\intercal \sigma_n = 0 \ \forall n' = 1, \ldots, N$. Together these restrictions suffice for condition (ii) of the claim to be met (with $\lambda_n = e^{(T-t)\sigma_n^\intercal \sigma_n}$ in particular).[28]

The two restrictions are met, for instance, by the dispersion matrix $\Sigma_4$ with respect to the second or third stock ($n = 2, 3$) as long as $\sigma_{n'1}\sigma_{21} = \sigma_{21}^2 + \sigma_{22}^2$ and $\sigma_{n'1}\sigma_{31} = \sigma_{31}^2 + \sigma_{33}^2$, for $n' \in \{1, 3\}$ and $n' \in \{1, 2\}$, respectively. We ought to have then $\sigma_{nk} \frac{\partial p_n(t)}{\partial \beta_k(t)} > 0$ for $k = 1, 2, 3$. Similarly, for the matrix $\Sigma_5$, the claim would apply on the second security if the terminal-period endowment is deterministic and $\sigma_{11}\sigma_{21} = \sigma_{21}^2 + \sigma_{22}^2$. In this case, $\sigma_{2k} \frac{\partial p_2(t)}{\partial \beta_k(t)} > 0$ for either of the two Brownian motions.

$$\Sigma_5 = \begin{pmatrix} \sigma_{11} & 0 \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \qquad \Sigma_6 = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

Of course, if the first stock were also correlated with the second Brownian component, as in the example $\Sigma_6$, the relevant restriction would read $\sigma_{21}(\sigma_{21} - \sigma_{11}) = \sigma_{22}(\sigma_{12} - \sigma_{22})$. In this case, the result would apply also on the first security if $\sigma_{11}(\sigma_{21} - \sigma_{11}) = \sigma_{12}(\sigma_{12} - \sigma_{22})$. That is, $\sigma_{nk} \frac{\partial p_n(t)}{\partial \beta_k(t)} > 0$ for $n, k \in \{1, 2\}$.

Regarding the last example above, it should be pointed out that the sufficient for condition (ii) restriction on the factor loadings may apply on both stocks only if the matrix $\Sigma_6$ is degenerate. In fact, $(\sigma_{n'} - \sigma_n)^\intercal \sigma_n = 0$ $\forall n' = 1, \ldots, N$ can hold for any risky security $n$ in the model, only if $\Sigma$ has

[28] Observe that $W\left(\mathcal{I}(t), \mathbf{x} + \sqrt{T-t}\sigma_n\right) = \sum_{n'=1}^N e^{(T-t)\sigma_{n'}^\intercal \sigma_n} e^{\mu_{n'} T + \sigma_{n'}^\intercal \left(\beta(t) + \sqrt{T-t}\mathbf{x}\right)} = e^{(T-t)s_n} \sum_{n'=1}^N e^{\mu_{n'} T + \sigma_{n'}^\intercal \left(\beta(t) + \sqrt{T-t}\mathbf{x}\right)} = e^{(T-t)\sigma_n^\intercal \sigma_n} W\left(\mathcal{I}(t), \mathbf{x}\right)$.

identical rows, being of the form $\Sigma = (\sigma_1 \mathbf{e}, \ldots, \sigma_K \mathbf{e})$ where $\mathbf{e} = \sum_{n=1}^{N} \mathbf{e}_n$.[29] In this case, even when markets are potentially dynamically complete ($N = K$), they will be necessarily dynamically incomplete. As it applies now to each and every stock in the model, Claim 3.1 restricts each row of the Jacobian matrix of relative prices to be a multiple of the respective row of $\Sigma$. More precisely, we have

$$|J_p(t)| = |\Sigma| \prod_{n=1}^{K} \lambda_n^{\alpha-1} e^{\mu_n T + \sigma_n^{\intercal}\left(\beta(t) - \frac{(T-t)\sigma_n}{2}\right)}$$

but the factor loadings matrix $\Sigma$ is now singular.[30]

# 5 Dynamic Completeness

[31] In an Arrow-Debreu economy, the agents may shift consumption across states and time by trading a complete set of contingent claims. When they are constrained to trade a given set of securities, however, the market is said to be dynamically complete if the agents can still achieve any consumption allocation that would be feasible if there were instead a complete set of Arrow-Debreu contingent claims. Under continuous-time trading, when the information about the state of the world is revealed through a stochastic process, this may be possible by trading a given finite set of securities rapidly enough. In particular, when the underlying uncertainty is driven by Brownian motions, a necessary (yet by no means sufficient) condition for this to happen is that the securities market is potentially dynamically complete: i.e., the number of securities exceeds that of independent Brownian motions by at least one.[32]

[29] For arbitrary $n', n'' \in \{1, \ldots, N\}$, the requirement in the text can be written as follows $\sum_{k=1}^{K} \sigma_{n'k}(\sigma_{n''k} - \sigma_{n'k}) = 0 = \sum_{k=1}^{K} \sigma_{n''k}(\sigma_{n'k} - \sigma_{n''k})$. Put differently, $\sum_{k=1}^{K} (\sigma_{n'k} - \sigma_{n''k})^2 = 0$ or $\sigma_{n'k} = \sigma_{n''k}\ \forall k$.

[30] Of course, given Theorem 2, dynamic completeness can be ruled out immediately once it is observed that $\Sigma$ is singular.

[31] To facilitate the comparison with the exposition of Anderson and Raimondo [5], my notation in this section re-introduces the functional dependence upon the state space $\Omega$.

[32] When the underlying information process is not Brownian, the required number of securities may be larger.

Let, therefore, $N = K$. As is well-known, in the presence of a money market account, dynamic completeness is equivalent to the dispersion matrix of absolute prices $\left[\frac{\partial P_n(\omega,t)}{\partial \beta_k(\omega,t)}\right]_{(n,k)\in\{0,\ldots,K\}\times\{1,\ldots,K\}}$ having almost everywhere rank equal to $K$, the number of the risk sources (see, for example, Sections 4.1-4.4 and Theorem 5.6 in Nielsen [31]). Here, one of the securities, the zero-coupon bond, is itself a money market account (the corresponding self-financing strategy being to hold $1/P_0(\omega, t)$ units of the bond) while its price is strictly positive everywhere on $\Omega \times [0, T]$. Renormalizing, therefore, dynamic completeness is equivalent to the $K \times K$ dispersion matrix of relative prices $J_p(\omega, t)$ having almost everywhere on $\Omega \times [0, T)$ rank equal to $K$, the dimension of the Brownian process. In the economy I examine, this is in turn equivalent to the matrix of factor loadings being nondegenerate.

**Theorem 2** *Let the securities market be potentially dynamically complete ($N = K$). The following are equivalent.*

*(i) The market is in fact dynamically complete.*

*(ii) $\Sigma$ is nonsingular.*

As a desirable feature of the economic environment under study, the nondegeneracy of the dispersion matrix $\Sigma$ was introduced in the literature by Harrison and Kreps [20] in order to ensure that the observable payoffs' process $Y$ generates the (generally) unobservable Brownian filtration, the true underlying informational structure.[33] In this sense, the nonsingularity of $\Sigma$ has been since regarded as fundamental and, given that it is equivalent to dynamic completeness when time is discrete, often conjectured to be related to dynamic completeness also in continuous time. A conjecture that, as Theorem 2 establishes formally, is correct in the strongest sense.

As a property, condition (ii) of the theorem depends only on the structure of the terminal dividends of the securities, leaving no role for the other

[33] I am referring to Proposition 1 in Harrison and Kreps [20] which allows also for $\Sigma$ to be stochastic so that the $N$-dimensional Ito process $Y$ could be given by $\mathrm{d}Y = \mu(Y(t), t)\,\mathrm{d}t + \Sigma(Y(t), t)\,\mathrm{d}\beta(\omega, t)$ with $Y(0) = \mathbf{0}$. As long as $\Sigma(\mathbf{x}, t)$ is nonsingular at every $(\mathbf{x}, t) \in \mathbb{R}^N \times [0, T]$, the Brownian filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$ is generated by $\{Y_t : 0 \leq t \leq T\}$.

economic primitives - in particular, the utility function of the representative agent or her endowment. It is also easily verified (by checking whether $|\Sigma| \neq 0$) and generically satisfied (within $\mathbb{R}^{2K}$, the set of points corresponding to singular square matrices is of zero-measure). And combining generic validity with universal verifiability is quite rare in the literature. In most generic results on dynamic completeness, the corresponding condition is shown to hold except for a small set of the primitive parameters, being nevertheless difficult (if not impossible in some cases) to establish whether it does for particular values of these parameters.

Under the terminal dividend specification in (4), the partial derivative with respect to the current realization of the $k$th Brownian component at the node $(\omega, t)$ and for the realization $\mathbf{x} \in \mathbb{R}^K$ of the Brownian increment $\beta(\omega, T) - \beta(\omega, t)$ is given by $\frac{\partial D_n(\mathcal{I}(\omega,t),\mathbf{x})}{\partial \beta_k(\omega,t)} = \sigma_{nk} D_n(\mathcal{I}(\omega, t), \mathbf{x})$. As a consequence, the $K \times K$ Jacobian matrix of terminal dividends

$$J_D(\mathcal{I}(\omega, t), \mathbf{x}) = \begin{bmatrix} \nabla_{\beta(\omega,t)} D_1(\mathcal{I}(\omega, t), \mathbf{x})^\intercal \\ \cdot \\ \cdot \\ \nabla_{\beta(\omega,t)} D_K(\mathcal{I}(\omega, t), \mathbf{x})^\intercal \end{bmatrix}$$

is constructed by multiplying each row of $\Sigma$ by the corresponding terminal dividend:

$$J_D(\mathcal{I}(\omega, t), \mathbf{x}) = [\sigma_n^\intercal D_n(\mathcal{I}(\omega, t), \mathbf{x})]_{n=1,\ldots,K} \tag{24}$$

It follows, therefore, that the nondegeneracy condition (ii) of the theorem is equivalent to requiring that $J_D(\beta(\omega, t), \mathbf{x})$ be of full rank at every node $(\omega, t)$ and for every realization $\mathbf{x} \in \mathbb{R}^K$ of the Brownian increment $\beta(\omega, T) - \beta(\omega, t)$. In other words, that $D_1(\mathcal{I}(\omega, t), \mathbf{x}), \ldots, D_K(\mathcal{I}(\omega, t), \mathbf{x})$ be locally linearly independent at every $(\omega, t, \mathbf{x}) \in \Omega \times [0, T) \times \mathbb{R}^K$.[34]

Obviously, if there are just enough securities for potential dynamic com-

[34]Recall that, if the matrix $\widetilde{A}$ results from multiplying a row of the square matrix $A$ by the number $\lambda$, then $|\widetilde{A}| = \lambda |A|$. In our case, $|J_D(\mathcal{I}(\omega, t), \mathbf{x})| = |\Sigma| \prod_{n=1}^K D_n(\mathcal{I}(\omega, t), \mathbf{x})$ with $\prod_{n=1}^K D_n(\mathcal{I}(\omega, t), \mathbf{x}) > 0$.

pleteness, some form of linear independence amongst the securities' dividends is necessary for dynamic completeness of the Arrow-Debreu securities prices. In this sense, some form of linear independence amongst the dividends is (at least implicitly) assumed in any paper within the realm of the continuous-time finance literature that deals with the case of potentially dynamically complete markets. Of course, not all of these papers have lump terminal dividends and not all present the corresponding form of dividend linear independence explicitly. In fact, to the best of my knowledge, the only one that does both is Anderson and Raimondo [5], and their linear independence assumption is equivalent to the one I present here, for the corresponding setting.

Anderson and Raimondo [5] prove existence of equilibrium in a continuous-time securities market setting that embeds the one I examine. They also study a single consumption good, pure exchange economy in which the information- and time-structure for trade and consumption are exactly as here. Yet, their typical security may pay dividends even during the intermediate period, their economy has many heterogenous agents, while they allow for time- as well as state-dependence in the dividends, endowments, and instantaneous utilities (as long as the latter dependence obtains only through the realizations of the Brownian process.).

Their securities market is potentially dynamically complete for they introduce $K+1$ securities. In state $\omega$, their typical security pays a dividend (measured in units of consumption) at some flow rate $d_n(\mathcal{I}(\omega,t))$ at times $t \in [0,T)$ and a lump amount $D_n(\mathcal{I}(\omega,T))$ at the terminal date. Their typical agent is endowed with the consumption good at some flow rate $e_i(\mathcal{I}(\omega,t))$ at times $t \in [0,T)$ and a lump amount $\rho_i(\mathcal{I}(\omega,T))$ at the end. Her preferences over consumption are given by a von Neumann-Morgenstern utility function $U_i$, such as the one I have considered in (3), in which the instantaneous utility functions $v_i$ and $u_i$ are defined on her measurable consumption process $c_i : [0,T] \times \Omega \mapsto \mathbb{R}_{++}$ but also on the process $\mathcal{I}$.

The authors take the functions that apply on flows, $d_n, e_i : \mathbb{R}^K \times [0,T) \mapsto \mathbb{R}_+$ and $v_i : \mathbb{R}_+ \times \mathbb{R}^K \times [0,T) \mapsto \mathbb{R} \cup \{-\infty\}$, to be analytic.[35] Regarding

[35] A function is said to be analytic if, at every point in its domain, there exists a power

the ones that apply on lump amounts, $D_n, \rho_i : \mathbb{R}^K \times \{T\} \mapsto \mathbb{R}_+$ and $u_i : \mathbb{R}_+ \times \mathbb{R}^K \times \{T\} \mapsto \mathbb{R} \cup \{-\infty\}$, the first two are assumed to be continuous and the third twice continuously differentiable, all almost everywhere on their respective domains. In addition, the functions $v_i$ and $u_i$ are required to satisfy certain standard regularity conditions.

More importantly for the purposes of my analysis, Anderson and Raimondo [5] assume the following nondegeneracy condition on the terminal dividends: there exist (i) an open set $V \subset \mathbb{R}^K$ such that the terminal dividend of security 0 is positive if the terminal-date realization of the Brownian vector falls within this set ($D_0(\mathbf{y}, T) > 0 \ \forall \mathbf{y} \in V$) and (ii) some terminal-date Brownian realization $\mathbf{y}^* \in V$ such that the $K \times K$ Jacobian matrix $J_{D_0}(\mathbf{y}, T)$ has full rank at $(\mathbf{y}^*, T)$.

$$J_{D_0}(\mathbf{y}, T) = \begin{bmatrix} \nabla_{\mathbf{y}} \left( \frac{D_1(\mathbf{y},T)}{D_0(\mathbf{y},T)} \right)^{\intercal} \\ . \\ . \\ \nabla_{\mathbf{y}} \left( \frac{D_K(\mathbf{y},T)}{D_0(\mathbf{y},T)} \right)^{\intercal} \end{bmatrix} \qquad J_D(\mathbf{y}, T) = \begin{bmatrix} \nabla_{\mathbf{y}} D_1(\mathbf{y}, T)^{\intercal} \\ . \\ . \\ \nabla_{\mathbf{y}} D_K(\mathbf{y}, T)^{\intercal} \end{bmatrix}$$

As it turns out, this exogenous assumption is sufficient for their equilibrium pricing process to be dynamically complete. As it happens, if the security 0 is a zero-coupon bond ($D_0(\mathcal{I}(\omega, t)) = 0$ for $t < T$ and $D_0(\mathcal{I}(\omega, T)) = 1$), this rank condition is nothing but the requirement that the Jacobian matrix $J_D(\mathbf{y}, T)$ is nonsingular at $(\mathbf{y}^*, T)$.

Needless to say, the economy I study in the present paper is a special case of the one just described. For it obtains by restricting the functions $d_n$ and $e_i$ to be, respectively, zero and deterministic (hence, either trivially analytic) while all terminal dividends in (4) are certainly continuous. Moreover, all conventional state-independent utility functions satisfy the conditions Anderson and Raimondo impose on $v_i$ and $u_i$. In fact, the authors themselves use the setting I analyze here as their main example (see their Section 3). It should be expected, therefore, that condition (ii) of Theorem 2 coincides with their nondegeneracy condition for the case in which security 0 is a

series which converges to the function on an open set containing the point.

zero-coupon bond.

Indeed, under the terminal dividend specification in (4), $D_n(\mathcal{I}(\omega,T)) = e^{\mu_n T + \sigma_n^{\intercal}\beta(\omega,T)}$ implies that the typical entry of the matrix $J_D(\mathbf{y},T)$ reads $\sigma_{nk} D_n(\mathbf{y},T)$, which is identical to the one in (24) once we define $\beta(\omega,T) = \mathbf{y} = \beta(\omega,t) + \mathbf{x}$. Hence, requiring the existence of some open set $V \subseteq \mathbb{R}^K$ and a point $\mathbf{y}^* \in V$ such that $J_D(\mathbf{y}^*,T)$ is nonsingular is equivalent to requiring that, conditional on the current realization $\beta(\omega,t)$, there exists some open set $V_{\beta(\omega,t)} \subseteq \mathbb{R}^K$ and a point $\mathbf{x}^* \in V_{\beta(\omega,t)}$ such that $J_D(\mathcal{I}(\omega,t),\mathbf{x}^*)$ is nonsingular. In my formulation, however, when valid, this nondegeneracy condition remains so universally rather than at some point of an open set. In view of (24), the nondegeneracy of the Jacobian matrix of terminal dividends is globally equivalent to the nondegeneracy of $\Sigma$, a matrix of constants.

Of course, my analysis considers an economy with a representative, not many heterogenous agents. Yet, this does not matter with respect to a condition for dynamic completeness that is imposed on the structure of the securities dividends only. It is well-known that, under dynamic completeness, the financial equilibrium must be Pareto optimal. Which, in an economy with many agents, requires in turn the existence of a (constant) vector of utility weights such that, at each node $(\omega,t)$, the equilibrium consumptions maximize the weighted sum of the utilities of the agents.

This weighted sum being the utility of the representative agent, Theorem 2 applies immediately, because it is not concerned with the functional form of the weight sum (or that of the social endowment). Its claim remains in force even with many heterogenous agents. In this sense, it is not surprising that Theorem 2 gives the same sufficient condition for dynamic completeness as Anderson and Raimondo do when one of their securities is a zero-coupon bond. What might be surprising perhaps, is that the theorem, in the important special case of the Anderson and Raimondo setup that I examine, complements their nondegeneracy hypothesis by rendering it also necessary for dynamic completeness.

## 6 Concluding Remarks

The main aim of this paper is to make the point that asset-price dynamics with respect to the underlying fundamental risk, even in the simple economy under study, is complex to the extent that assertions about the direction of asset-price movements cannot be supported, except for particular situations, even when the dividend of the security is not correlated with the risk source in question. In presenting this thesis, my strategy has been to find specifications for the economic primitives under which the sign of the correlation between the relative price of the typical security and the typical underlying Brownian motion remains unambiguous throughout the stochastic domain.

By establishing that, as a norm, asset prices *are* correlated with an underlying risk source even when payoffs are not, my findings indicate that asset-price dynamics are much richer than one is led to expect at first glance, armed with basic economic intuition. By showing, on the other hand, that it is by no means straightforward to identify settings in which the sign of this correlation remains constant, they attest to the complexity of these dynamics. Together, richness and complexity suggest a tumultuous financial world, even in the benchmark model of a fully rational, price-taking, representative agent.

Even though my focus has been purely theoretical, it is important that my results apply on the entire family of state-independent utility functions that are monotone in risk-aversion. My formulae, moreover, can be calculated numerically for any set of the model parameters. Which is relevant since my findings are of consequence also for applications. The fact that the equilibrium relative prices of assets and asset returns should be correlated, even when their underlying dividends are independent, has significant implications for empirical asset-pricing. In particular, it raises questions about the large body of work that focusses on partial-equilibrium analysis, treating a small number of securities in isolation from the rest of the market or modeling the equilibrium price process of an asset as a relation that depends only on those risk sources that directly affect its payoff.

Of course, my results do not extend beyond state-independent utility functions. Yet, within the context of general equilibrium analysis, this restriction should not be taken at face value. One of the reasons that state-dependence appears natural in some models is because they are partial equilibrium studies. If a significant portion of household wealth is held on an asset that is not included in the model, changes in the value of this asset induce wealth effects that alter the agents' willingness to hold those assets the model does include. As a consequence, value changes in the omitted asset seem to be instances of state-dependent felicity.

In a general equilibrium model, however, which includes all relevant assets, this kind of state-dependence would disappear, rendering without loss of generality that the utility function is exogenously specified. In this sense, the real limitation of my analysis lies in the dividend specification, which can only be a geometric Brownian motion. Even though a widely-used specification, especially in empirical continuous-time finance, it does nonetheless constraint the validity of my results. For my main proofs, at some point or another, all exploit the symmetry of the Normal distribution.

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# Appendices

## A Preliminary Results

**Lemma A.1** *Given a twice-differentiable function $H : \mathbb{R}^{K+1} \mapsto \mathbb{R}$ and an open interval $I_\beta(\epsilon) = (\beta - \epsilon, \beta + \epsilon)$ around the point $\beta \in \mathbb{R}$, suppose that $G : I_\beta(\epsilon) \mapsto \mathbb{R}$ given by*

$$G\left(\widetilde{\beta}\right) = \int_{\mathbb{R}^K} H\left(\widetilde{\beta}, \mathbf{x}\right) d\Phi(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}\left[H\left(\widetilde{\beta}, \mathbf{x}\right)\right]$$

*is well-defined and that*

$$s_\beta := \sup_{\widetilde{\beta}\in[\beta-\epsilon,\beta+\epsilon]} \sup_{\mathbf{x}\in\mathbb{R}^K} \left|\frac{\partial^2}{\partial\beta^2} H\left(\widetilde{\beta}, \mathbf{x}\right)\right| \prod_{k=1}^{K} \max\left\{1, x_k^2\right\} \exp\left(-\frac{\mathbf{x}^\intercal \mathbf{x}}{2}\right) < +\infty$$

*Then $G$ is differentiable at $\beta$ with $G'(\beta) = \mathbb{E}_{\mathbf{x}}\left[\frac{\partial}{\partial\beta} H(\beta, \mathbf{x})\right]$.*

**Proof.** For any $z \in \mathbb{R} \setminus \{0\} : |z| < \epsilon$, we have

$$\begin{aligned}
& \left| \frac{G(\beta + z) - G(\beta)}{z} - \int_{\mathbb{R}^K} \frac{\partial}{\partial \beta} H(\beta, \mathbf{x}) \, \mathrm{d}\Phi(\mathbf{x}) \right| \\
= & \left| \int_{\mathbb{R}^K} \left( \frac{H(\beta + z, \mathbf{x}) - H(\beta, \mathbf{x})}{z} - \frac{\partial}{\partial \beta} H(\beta, \mathbf{x}) \right) \mathrm{d}\Phi(\mathbf{x}) \right| \\
\leq & \int_{\mathbb{R}^K} \left| \frac{H(\beta + z, \mathbf{x}) - H(\beta, \mathbf{x})}{z} - \frac{\partial}{\partial \beta} H(\beta, \mathbf{x}) \right| \mathrm{d}\Phi(\mathbf{x}) \\
= & \int_{\mathbb{R}^K} \left| \frac{\partial}{\partial \beta} H(\beta +_{\mathbf{x}} z, \mathbf{x}) - \frac{\partial}{\partial \beta} H(\beta, \mathbf{x}) \right| \mathrm{d}\Phi(\mathbf{x}) \quad \text{for some } \gamma_{\mathbf{x}} \in (0,1) \\
= & \int_{\mathbb{R}^K} \left| z \gamma_{\mathbf{x}} \frac{\partial^2}{\partial \beta^2} H(\beta + \delta_{\mathbf{x}} \gamma_{\mathbf{x}} z, \mathbf{x}) \right| \mathrm{d}\Phi(\mathbf{x}) \quad \text{for some } \delta_{\mathbf{x}} \in (0,1) \\
< & \, |z| \int_{\mathbb{R}^K} \left| \frac{\partial^2}{\partial \beta^2} H(\beta + \delta_{\mathbf{x}} \gamma_{\mathbf{x}} z, \mathbf{x}) \right| \mathrm{d}\Phi(\mathbf{x}) \leq \frac{|z| \, s_\beta}{\sqrt{(2\pi)^K}} \int_{\mathbb{R}^K} \frac{\mathrm{d}\mathbf{x}}{\prod_{k=1}^{K} \max\left\{1, x_k^2\right\}}
\end{aligned}$$

where the second and third equalities are due to the mean-value theorem while the two inequalities follow from $|\gamma_{\mathbf{x}}| < 1$ and by hypothesis, respectively. Yet, the $x_k$'s are independently distributed so that

$$\int_{\mathbb{R}^K} \prod_{k=1}^{K} \frac{\mathrm{d}\mathbf{x}}{\max\left\{1, x_k^2\right\}} = \prod_{k=1}^{K} \int_{\mathbb{R}} \frac{\mathrm{d}x_k}{\max\left\{1, x_k^2\right\}} = \prod_{k=1}^{K} \left( \int_{-1}^{1} \mathrm{d}x_k + 2 \int_{1}^{+\infty} x_k^{-2} \mathrm{d}x_k \right) = 4^K$$

and taking $|z| \to 0$ proves the claim. ■

The following is a well-known result.

**Lemma A.2** *Let* $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, I_K)$, $\theta \in \mathbb{R}^K$, *and* $g : \mathbb{R}^K \to \mathbb{R}$ *s.t.* $\mathbb{E}_{\mathbf{z}}\left[e^{\theta^\intercal \mathbf{z}} g(\mathbf{z})\right]$ *is well-defined. Then* $\mathbb{E}_{\mathbf{z}}\left[e^{\theta^\intercal \mathbf{z}} g(\mathbf{z})\right] = e^{\frac{\theta^\intercal \theta}{2}} \mathbb{E}_{\mathbf{z}}\left[g(\mathbf{z} + \theta)\right]$.

The next lemma will be used in establishing its antecedent.

**Lemma A.3** *Let* $\phi, \psi : \mathbb{R} \to \mathbb{R}$ *be twice-differentiable functions such that the following integrals are defined*

*(i)* $\int_{\mathbb{R}} \phi(z) \psi'(z) \, dz$ *and* $\int_{\mathbb{R}} \phi'(z) \psi(z) \, dz$

*(ii)* $\int_{-\infty}^{m} \phi(z) \psi'(z) \, dz$ *and* $\int_{-\infty}^{m} \phi'(z) \psi(z) \, dz$, *for some* $m \in \mathbb{R}$

*(iii)* $\int_l^{+\infty} \phi(z)\,\psi'(z)\,dz$*, and* $\int_l^{+\infty} \phi'(z)\,\psi(z)\,dz$*, for some* $l \in \mathbb{R}$*.*

*Then* $\int_{\mathbb{R}} \phi(z)\,\psi'(z)\,dz = \lim_{a\to+\infty} \phi(a)\,\psi(d) - \lim_{b\to-\infty} \phi(b)\,\psi(c) - \int_{\mathbb{R}} \phi'(z)\,\psi(z)\,dz$*.*

**Proof.** For the given $l, m \in \mathbb{R}$, we can write[36]

$$\int_{\mathbb{R}} \phi(z)\,\psi'(z)\,\mathrm{d}z = \int_{-\infty}^{m} \phi(z)\,\psi'(z)\,\mathrm{d}z + \int_{m}^{l} \phi(z)\,\psi'(z)\,\mathrm{d}z + \int_{l}^{+\infty} \phi(z)\,\psi'(z)\,\mathrm{d}z$$

Using standard integration-by-parts, the proper integral above becomes

$$\int_m^l \phi(z)\,\psi'(z)\,\mathrm{d}z = \phi(l)\,\psi(l) - \phi(m)\,\psi(m) - \int_m^l \phi'(z)\,\psi(z)\,\mathrm{d}z$$

while the two improper ones can be written as follows

$$\begin{aligned}
\int_{-\infty}^{m} \phi(z)\,\psi'(z)\,\mathrm{d}z &= \lim_{b\to-\infty} \int_b^m \phi(z)\,\psi'(z)\,\mathrm{d}z \\
&= \lim_{b\to-\infty} \left( \phi(m)\,\psi(m) - \phi(b)\,\psi(b) - \int_b^m \phi'(z)\,\psi(z)\,dz \right) \\
&= \phi(m)\,\psi(m) - \lim_{b\to-\infty} \phi(b)\,\psi(b) - \int_{-\infty}^{m} \phi'(z)\,\psi(z)\,\mathrm{d}z \\
\int_l^{+\infty} \phi(z)\,\psi'(z)\,\mathrm{d}z &= \lim_{a\to+\infty} \int_l^a \phi(z)\,\psi'(z)\,\mathrm{d}z \\
&= \lim_{a\to+\infty} \left( \phi(a)\,\psi(a) - \phi(l)\,\psi(l) - \int_l^K \phi'(z)\,\psi(z)\,\mathrm{d}z \right) \\
&= \lim_{a\to+\infty} \phi(a)\,\psi(a) - \phi(l)\,\psi(l) - \int_l^{+\infty} \phi'(z)\,\psi(z)\,\mathrm{d}z
\end{aligned}$$

i.e.,

$$\begin{aligned}
\int_{\mathbb{R}} \phi(z)\,\psi'(z)\,\mathrm{d}z &= \lim_{a\to+\infty} \phi(a)\,\psi(a) - \lim_{b\to-\infty} \phi(b)\,\psi(b) \\
&\quad - \left( \int_{-\infty}^{m} \phi'(z)\,\psi(z)\,\mathrm{d}z + \int_m^l \phi'(z)\,\psi(z)\,\mathrm{d}z + \int_l^{+\infty} \phi'(z)\,\psi(z)\,\mathrm{d}z \right) \\
&= \lim_{a\to+\infty} \phi(a)\,\psi(a) - \lim_{b\to-\infty} \phi(b)\,\psi(b) - \int_{\mathbb{R}} \phi'(z)\,\psi(z)\,\mathrm{d}z
\end{aligned}$$

[36]Since the improper integrals $\int_{\mathbb{R}} \phi(z)\,\psi'(z)\,\mathrm{d}z$, $\int_{-\infty}^{m} \phi(z)\,\psi'(z)\,\mathrm{d}z$, and $\int_l^{+\infty} \phi(z)\,\psi'(z)\,\mathrm{d}z$ are all defined, so is the proper one $\int_m^l \phi(z)\,\psi'(z)\,\mathrm{d}z$.

as required. ■

**Lemma A.4** *Let* $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, I_K)$, $\theta \in \mathbb{R}^K$, *and* $g : \mathbb{R}^K \mapsto \mathbb{R}$ *s.t. the following conditions are met.*

*(i)* $\mathbb{E}_{\mathbf{z}}\left[e^{\theta'\mathbf{z}}\frac{\partial g(\mathbf{z})}{\partial z_k}\right]$ *and* $\mathbb{E}_{\mathbf{z}}\left[z_k g(\mathbf{z}+\theta)\right]$ *are well-defined.*

*(ii) Given any* $\mathbf{z}_{-k} \in \mathbb{R}^{K-1}$, *Lemma A.3 applies on the functions* $\psi, \phi : \mathbb{R} \to \mathbb{R}$ *given by* $\psi(z_k) = g(z_k, \mathbf{z}_{-k})$ *and* $\phi(z_k) = e^{\theta^\intercal(z_k,\mathbf{z}_{-k}) - \frac{(z_k,\mathbf{z}_{-k})^\intercal(z_k,\mathbf{z}_{-k})}{2}}$ *while* $\lim_{z_k\to\pm\infty}\phi(z_k)\psi(z_k) = 0 \ \forall \mathbf{z}_{-k} \in \mathbb{R}^{K-1}$.

*Then* $\mathbb{E}_{\mathbf{z}}\left[e^{\theta'\mathbf{z}}\frac{\partial g(\mathbf{z})}{\partial z_k}\right] = e^{\frac{\theta^\intercal\theta}{2}}\mathbb{E}_{\mathbf{z}}\left[z_k g(\mathbf{z}+\theta)\right]$.

**Proof.** We have

$$\begin{aligned}
\mathbb{E}_{\mathbf{z}}\left[e^{\theta^\intercal\mathbf{z}}\frac{\partial g(\mathbf{z})}{\partial z_k}\right] &= \int_{\mathbb{R}^K} e^{\theta^\intercal\mathbf{z}}\frac{\partial g(\mathbf{z})}{\partial z_k}\mathrm{d}\Phi(\mathbf{z}) \\
&= \frac{1}{\sqrt{(2\pi)^K}}\int_{\mathbb{R}^{K-1}}\left(\int_{\mathbb{R}} e^{\theta^\intercal\mathbf{z}}\frac{\partial g(\mathbf{z})}{\partial z_k}e^{-\frac{z_k^2}{2}}dz_k\right)e^{-\frac{\sum_{i\neq k}z_i^2}{2}}\mathrm{d}\mathbf{z}_{-k}
\end{aligned}$$

By Lemma A.3, we can use integration by parts to simplify the integral in the brackets. Specifically, given $\mathbf{z}_{-k} \in \mathbb{R}^{K-1}$ and the functions $\phi : \mathbb{R} \to \mathbb{R}_+$ and $\psi : \mathbb{R} \to \mathbb{R}$ as in the proposition, we have

$$\begin{aligned}
\int_{\mathbb{R}} e^{\theta^\intercal\mathbf{z}}\frac{\partial g(\mathbf{z})}{\partial z_k}e^{-\frac{z_k^2}{2}}\mathrm{d}z_k &= \int_{\mathbb{R}}\phi(z_k)\psi'(z_k)\mathrm{d}z_k \\
&= \lim_{a\to+\infty}\phi(a)\psi(a) - \lim_{b\to-\infty}\phi(b)\psi(b) - \int_{\mathbb{R}}\phi'(z)\psi(z)\mathrm{d}z \\
&= \left(\lim_{z_k\to+\infty}e^{\theta^\intercal\mathbf{z}-\frac{z_k^2}{2}}g(\mathbf{z}) - \lim_{z_k\to-\infty}e^{\theta^\intercal\mathbf{z}-\frac{z_k^2}{2}}g(\mathbf{z})\right) - \int_{\mathbb{R}}(\theta_k - z_k)g(\mathbf{z})e^{\theta^\intercal\mathbf{z}-\frac{z_k^2}{2}}\mathrm{d}z_k \\
&= \int_{\mathbb{R}}(z_k - \theta_k)g(\mathbf{z})e^{\theta^\intercal\mathbf{z}-\frac{z_k^2}{2}}\mathrm{d}z_k
\end{aligned}$$

Integrating now over $\mathbf{z}_{-k} \in \mathbb{R}^{K-1}$ gives

$$\begin{aligned}
& \int_{\mathbb{R}^{K-1}} \left( \int_{\mathbb{R}} e^{\theta^\intercal \mathbf{z}} \frac{\partial g(\mathbf{z})}{\partial z_k} e^{-\frac{z_k^2}{2}} \mathrm{d}z_k \right) e^{-\frac{\sum_{i \neq k} z_i^2}{2}} \mathrm{d}\mathbf{z}_{-k} \\
= & \int_{\mathbb{R}^{K-1}} \left( \int_{\mathbb{R}} (z_k - \theta_k) g(\mathbf{z}) e^{\theta^\intercal \mathbf{z} - \frac{z_k^2}{2}} \mathrm{d}z_k \right) e^{-\frac{\sum_{i \neq k} z_i^2}{2}} \mathrm{d}\mathbf{z}_{-k} \\
= & \; e^{\frac{\theta^\intercal \theta}{2}} \int_{\mathbb{R}^K} (z_k - \theta_k) g(\mathbf{z}) e^{-\frac{\sum_i (z_i - \theta_i)^2}{2}} \mathrm{d}\mathbf{z} \\
= & \; e^{\frac{\theta^\intercal \theta}{2}} \int_{\mathbb{R}^K} (z_k - \theta_k) g(\mathbf{z}) e^{-\frac{(\mathbf{z}-\theta)^\intercal (\mathbf{z}-\theta)}{2}} \mathrm{d}\mathbf{z} = e^{\frac{\theta^\intercal \theta}{2}} \int_{\mathbb{R}^K} z_k g(\mathbf{z} + \theta) e^{-\frac{\mathbf{z}^\intercal \mathbf{z}}{2}} \mathrm{d}\mathbf{z}
\end{aligned}$$

and the result follows immediately. ■

**Lemma A.5** *Let $S \subseteq \mathbb{R}^n$ be of non-zero Lebesgue measure and such that $S^2$ is symmetric around the origin.*[37] *Suppose also that*

*(i)* $g : S^2 \mapsto \mathbb{R}_+$ *is symmetric - i.e., $g(\mathbf{x}, \mathbf{y}) = g(\mathbf{y}, \mathbf{x})$ - everywhere on its domain except for sets of measure zero,*[38]

*(ii)* $f : S^2 \mapsto \mathbb{R}$ *is such that $f(\mathbf{x}, \mathbf{y}) + f(\mathbf{y}, \mathbf{x}) \geq 0$ everywhere on its domain except for sets of measure zero, and*

*(iii)* $(gf)(\cdot)$ *is Lebesgue-integrable over $S^2$.*

*Then*

$$\int_{S^2} g(\mathbf{x}, \mathbf{y}) f(\mathbf{x}, \mathbf{y}) \, d(\mathbf{x}, \mathbf{y}) \geq 0$$

*with strict inequality iff $g(\mathbf{x}, \mathbf{y}) [f(\mathbf{x}, \mathbf{y}) + f(\mathbf{y}, \mathbf{x})] \neq 0$ on a subset of $S^2$ of non-zero measure.*

**Proof.** Since $(gf)(\cdot)$ is integrable, by the Fubini-Tonelli theorem, the integral in question can be written as an iterated one:

$$\int_{\mathrm{S}^2} g(\mathbf{x}, \mathbf{y}) f(\mathbf{x}, \mathbf{y}) \, \mathrm{d}(\mathbf{x}, \mathbf{y}) = \int_{\mathrm{S}} \left( \int_{\mathrm{S}} g(\mathbf{x}, \mathbf{y}) f(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\mathbf{y} \right) \mathrm{d}\mathbf{x}$$

[37] This is to say that the relation $R(\mathbf{x}, \mathbf{y}) := \langle (\mathbf{x}, \mathbf{y}) \in \mathrm{S}^2 \rangle \subseteq \mathbb{R}^{2n}$ is symmetric.

[38] The lemma holds, more generally, if $g$ is symmetric almost everywhere.

and, by re-naming the variables of integration, we can write it also as

$$\begin{aligned}\int_{\mathrm{S}^2} g(\mathbf{x},\mathbf{y}) f(\mathbf{x},\mathbf{y})\,\mathrm{d}(\mathbf{x},\mathbf{y}) &= \int_{\mathrm{S}^2} g(\mathbf{y},\mathbf{x}) f(\mathbf{y},\mathbf{x})\,\mathrm{d}(\mathbf{y},\mathbf{x}) \\ &= \int_{\mathrm{S}} \left( \int_{\mathrm{S}} g(\mathbf{y},\mathbf{x}) f(\mathbf{y},\mathbf{x})\,\mathrm{d}\mathbf{y} \right) \mathrm{d}\mathbf{x}\end{aligned}$$

Hence,

$$\begin{aligned}& 2\int_{\mathrm{S}^2} g(\mathbf{x},\mathbf{y}) f(\mathbf{x},\mathbf{y})\,\mathrm{d}(\mathbf{x},\mathbf{y}) \\ =\ & \int_{\mathrm{S}} \left( \int_{\mathrm{S}} g(\mathbf{x},\mathbf{y}) f(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} \right) \mathrm{d}\mathbf{x} + \int_{\mathrm{S}} \left( \int_{\mathrm{S}} g(\mathbf{y},\mathbf{x}) f(\mathbf{y},\mathbf{x})\,\mathrm{d}\mathbf{y} \right) \mathrm{d}\mathbf{x} \\ =\ & \int_{\mathrm{S}} \left( \int_{\mathrm{S}} g(\mathbf{x},\mathbf{y}) \left[ f(\mathbf{x},\mathbf{y}) + f(\mathbf{y},\mathbf{x}) \right] \mathrm{d}\mathbf{y} \right) \mathrm{d}\mathbf{x} \geq 0\end{aligned}$$

Obviously, the inequality is strict iff $g(\mathbf{x},\mathbf{y}) \left[ f(\mathbf{x},\mathbf{y}) + f(\mathbf{y},\mathbf{x}) \right] \neq 0$ on a subset of $\mathrm{S}^2$ of positive measure. ■

**Lemma A.6** *Let the random vector* $\mathbf{x} \in \mathbb{R}^K$ *and the function* $g : \mathbb{R}^K \mapsto \mathbb{R}$ *be s.t.* $\mathbb{E}_{\mathbf{x}}[g(\mathbf{x})]$ *and* $\mathbb{E}_{\mathbf{x}}[x_k g(\mathbf{x})]$ *are well-defined, with* $\mathbb{E}_{\mathbf{x}}[g(\mathbf{x})] \neq 0$. *Suppose also that* $f : \mathbb{R} \mapsto \mathbb{R}$ *is given by* $f(y_k) = \mathbb{E}_{\mathbf{x}}[(y_k - x_k) g(\mathbf{x})]$. *Then,*

$$\exists y_k^0 \in \mathbb{R} : \ \left( y_k - y_k^0 \right) f(y_k)\, \mathbb{E}_{\mathbf{x}}[g(\mathbf{x})] > 0 \quad \forall y_k \in \mathbb{R} \setminus \{y_k^0\}$$

**Proof.** Given that $\mathbb{E}_{\mathbf{x}}[g(\mathbf{x})] \neq 0$, we can write

$$f(y_k) = \mathbb{E}_{\mathbf{x}}[g(\mathbf{x})] \left( y_k - \frac{\mathbb{E}_{\mathbf{x}}[x_k g(\mathbf{x})]}{\mathbb{E}_{\mathbf{x}}[g(\mathbf{x})]} \right)$$

and it suffices to define $y_k^0 = \mathbb{E}_{\mathbf{x}}[x_k g(\mathbf{x})] / \mathbb{E}_{\mathbf{x}}[g(\mathbf{x})]$. ■

# B Comonotonicity and Covariance

For a set $S$ and an algebra $\sigma$ on $S$, let $B(S,\mathbb{R})$ be the set of bounded $\sigma$-measurable functions $S \mapsto \mathbb{R}$. Two random variables $g, f \in B(S,\mathbb{R})$ are said to be <u>comonotonic</u> if

$$\left[ g(\mathbf{x}) - g(\mathbf{y}) \right] \left[ f(\mathbf{x}) - f(\mathbf{y}) \right] \geq 0 \quad \forall \mathbf{x}, \mathbf{y} \in S$$

They are strictly comonotonic if the inequality is strict whenever $\mathbf{x} \neq \mathbf{y}$. The following result is borrowed from Chateauneuf et al. [10]. I present the relevant for my argument "only if" part of the proof.

**Lemma B.1** *$g, f \in B(S, \mathbb{R})$ are (strictly) comonotonic iff $Cov_\pi[g, f] \geq 0$ ($> 0$) for any prob. measure $\pi$ on $(S, \sigma)$.*

**Proof.** If $g$ and $f$ are comonotonic and $\pi$ a probability measure on $(S, \sigma)$,

$$\begin{aligned}
2\mathrm{Cov}_\pi[g, f] &= 2\left(\mathbb{E}_\pi[gf] - \mathbb{E}_\pi[g]\,\mathbb{E}_\pi[f]\right) \\
&= 2\left(\int_S g(\mathbf{x})\, f(\mathbf{x})\,\mathrm{d}\pi(\mathbf{x}) - \int_S g(\mathbf{y})\,\mathrm{d}\pi(\mathbf{y}) \int_S f(\mathbf{x})\,\mathrm{d}\pi(\mathbf{x})\right) \\
&= \int_S g(\mathbf{x})\, f(\mathbf{x})\,\mathrm{d}\pi(\mathbf{x}) + \int_S g(\mathbf{y})\, f(\mathbf{y})\,\mathrm{d}\pi(\mathbf{y}) \\
&\quad - \int_S g(\mathbf{y})\,\mathrm{d}\pi(\mathbf{y}) \int_S f(\mathbf{x})\,\mathrm{d}\pi(\mathbf{x}) - \int_S g(\mathbf{x})\,\mathrm{d}\pi(\mathbf{x}) \int_S f(\mathbf{y})\,\mathrm{d}\pi(\mathbf{y}) \\
&= \int_{S\times S} [g(\mathbf{x}) - g(\mathbf{y})]\,[f(\mathbf{x}) - f(\mathbf{y})]\,\mathrm{d}\pi(\mathbf{x})\,\mathrm{d}\pi(\mathbf{y}) \geq 0
\end{aligned}$$

where the third equality uses a change of the variables of integration. The validity of the claim when the comonotonicity is strict is obvious. ■

Regarding the application of this result in the main text, notice that $f$ and $g$ need not be bounded there. The boundedness condition guarantees that the integrals above exist for *any* prob. measure $\pi$ on $(S, \sigma)$. In the analysis of the asset-riskness effect, I fix $\mathbf{y} \in \mathbb{R}^{K-M}$ taking $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathrm{I}_M)$, $f : \mathbb{R}^M \mapsto \mathbb{R}_{++}$ and $g : \mathbb{R}^M \mapsto \mathbb{R}_{--}$ with $f(\mathbf{z}) = e^{\mu_n T + \sigma_n^\intercal\left(\beta(t) + \sqrt{T-t}\mathbf{z}\right)}$ and $g(\mathbf{z}) = u''(W(\mathcal{I}(t), (\mathbf{z}, \mathbf{y})))$. The relevant expectations are well-defined even though $f$ and $g$ are, respectively, not and not necessarily bounded. The strict comonotonicity between $f$ and $g$ is due to non-increasing absolute risk aversion, $r_A'(\cdot) \leq 0$. For this requires that $u'''(\cdot) > 0$ which in turn suffices since, other things being equal, $W(\mathcal{I}(t), (\mathbf{z}, \mathbf{y}))$ in (11) is strictly increasing in $f(\mathbf{z})$, the realization of the terminal dividend.

# C Proofs of the Results in the Text

This section presents the proofs for the various results in the paper. To keep notation simple, I will display neither the node $(\omega, t)$ of the Brownian filtration nor the process $\mathcal{I}$ as arguments in the relevant functions. Notice also that, even though not shown again for notational parsimony, all expectations are supposed to be conditional on the current filtration $\mathcal{F}_t$.

**Theorem 1**

Take $(n,k) \in \{1,...,N\} \times \{1,...,K\}$ and consider (9). $\frac{\partial p_n}{\partial \beta_k}$ and $\frac{\partial P_0}{\partial \beta_k}$ apply the partial-derivative operator $\frac{\partial}{\partial \beta_k}$ on $P_n = \mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)e^{\mu_n T + \sigma_n^\intercal\left(\beta + \sqrt{T-t}\mathbf{x}\right)}\right]$ and $P_0 = \mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)\right]$, respectively. Lemma A.1 guarantees that this operator commutes with the expectations operator in this case. As a result, the partial-derivative terms on the right-hand side of (9) may be written as follows

$$\begin{aligned}
\frac{\partial P_n}{\partial \beta_k} &= \mathbb{E}_{\mathbf{x}}\left[\frac{\partial}{\partial \beta_k}\left(u'\left(W\left(\mathbf{x}\right)\right)e^{\mu_n T + \sigma_n^\intercal\left(\beta + \sqrt{T-t}\mathbf{x}\right)}\right)\right] \\
&= \sigma_{nk}\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)e^{\mu_n T + \sigma_n^\intercal\left(\beta + \sqrt{T-t}\mathbf{x}\right)}\right] \\
&\quad + \mathbb{E}_{\mathbf{x}}\left[u''\left(W\left(\mathbf{x}\right)\right)e^{\mu_n T + \sigma_n^\intercal\left(\beta + \sqrt{T-t}\mathbf{x}\right)}\frac{\partial W\left(\mathbf{x}\right)}{\partial \beta_k}\right] \\
\frac{\partial P_0}{\partial \beta_k} &= \mathbb{E}_{\mathbf{x}}\left[u''\left(W\left(\mathbf{x}\right)\right)\frac{\partial W\left(\mathbf{x}\right)}{\partial \beta_k}\right]
\end{aligned}$$

Using Lemma A.2, moreover, we get

$$\begin{aligned}
p_n &= e^{\mu_n T + \sigma_n^\intercal\left(\beta + \frac{(T-t)}{2}\sigma_n\right)}\frac{\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x} + \sqrt{T-t}\sigma_n\right)\right)\right]}{\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)\right]} \\
\frac{\partial P_n}{\partial \beta_k} &= e^{\mu_n T + \sigma_n^\intercal\left(\beta + \frac{(T-t)}{2}\sigma_n\right)}\left(\begin{array}{c}\sigma_{nk}\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x} + \sqrt{T-t}\sigma_n\right)\right)\right] \\ + \mathbb{E}_{\mathbf{x}}\left[u''\left(W\left(\mathbf{x} + \sqrt{T-t}\sigma_n\right)\right)\frac{\partial W\left(\mathbf{x} + \sqrt{T-t}\sigma_n\right)}{\partial \beta_k}\right]\end{array}\right)
\end{aligned}$$

Combining, therefore, these relations gives

$$\begin{aligned}
\frac{P_0^2}{e^{\mu_n T+\sigma_n^\intercal \beta}}\frac{\partial p_n}{\partial \beta_k} &= \sigma_{nk}\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{x}}\right]\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)\right] \\
&\quad +\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)\right]\mathbb{E}_{\mathbf{x}}\left[u''\left(W\left(\mathbf{x}\right)\right)e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{x}}\frac{\partial W\left(\mathbf{x}\right)}{\partial \beta_k}\right] \\
&\quad -\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{x}}\right]\mathbb{E}_{\mathbf{x}}\left[u''\left(W\left(\mathbf{x}\right)\right)\frac{\partial W\left(\mathbf{x}\right)}{\partial \beta_k}\right] \\
&= \sigma_{nk}\mathbb{E}_{\mathbf{y}}\left[u'\left(W\left(\mathbf{y}\right)\right)e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{y}}\right]\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)\right] \qquad (25) \\
&\quad +\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)\right]\mathbb{E}_{\mathbf{y}}\left[u''\left(W\left(\mathbf{y}\right)\right)e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{y}}\frac{\partial W\left(\mathbf{y}\right)}{\partial \beta_k}\right] \\
&\quad -\mathbb{E}_{\mathbf{y}}\left[u'\left(W\left(\mathbf{y}\right)\right)e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{y}}\right]\mathbb{E}_{\mathbf{x}}\left[u''\left(W\left(\mathbf{x}\right)\right)\frac{\partial W\left(\mathbf{x}\right)}{\partial \beta_k}\right]
\end{aligned}$$

the second equality using a re-naming of variables of integration with $\mathbf{y},\mathbf{x}\sim$ i.i.d. $\mathcal{N}\left(\mathbf{0},\mathrm{I}_K\right)$. For the terminal-period wealth, on the other hand, we have

$$\begin{aligned}
\frac{\partial W\left(\mathbf{x}\right)}{\partial \beta_k} &= \frac{\partial}{\partial \beta_k}\left(\rho\left(\beta+\sqrt{T-t}\mathbf{x}\right)+\sum_{i=1}^N e^{\mu_i T+\sigma_i^\intercal\left(\beta+\sqrt{T-t}\mathbf{x}\right)}\right) \\
&= \frac{\partial \rho\left(\beta+\sqrt{T-t}\mathbf{x}\right)}{\partial \beta_k}+\sum_{i=1}^N \sigma_{ik}e^{\mu_i T+\sigma_i^\intercal\left(\beta+\sqrt{T-t}\mathbf{x}\right)} \qquad (26) \\
&= \frac{1}{\sqrt{T-t}}\frac{\partial}{\partial x_k}\left(\rho\left(\beta+\sqrt{T-t}\mathbf{x}\right)+\sum_{i=1}^N e^{\mu_i T+\sigma_i^\intercal\left(\beta+\sqrt{T-t}\mathbf{x}\right)}\right) \\
&= \frac{1}{\sqrt{T-t}}\frac{\partial W\left(\mathbf{x}\right)}{\partial x_k}
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{P_0^2}{e^{\mu_n T+\sigma_n^\intercal \beta_n}}\frac{\partial p_n}{\partial \beta_k} &= \sigma_{nk}\mathbb{E}_{\mathbf{y}}\left[u'\left(W\left(\mathbf{y}\right)\right)e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{y}}\right]\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)\right] \\
&\quad +\frac{1}{\sqrt{T-t}}\mathbb{E}_{\mathbf{y}}\left[e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{y}}\frac{\partial u'\left(W\left(\mathbf{y}\right)\right)}{\partial y_k}\right]\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)\right] \\
&\quad -\frac{1}{\sqrt{T-t}}\mathbb{E}_{\mathbf{x}}\left[\frac{\partial u'\left(W\left(\mathbf{x}\right)\right)}{\partial x_k}\right]\mathbb{E}_{\mathbf{y}}\left[u'\left(W\left(\mathbf{y}\right)\right)e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{y}}\right]
\end{aligned}$$

Apply now Lemma A.2 on the term $\mathbb{E}_{\mathbf{y}}\left[u'\left(W\left(\mathbf{y}\right)\right)e^{\sqrt{T-t}\sigma_n^{\intercal}\mathbf{y}}\right]$ and Lemma A.4 on each of $\mathbb{E}_{\mathbf{y}}\left[e^{\sqrt{T-t}\sigma_n^{\intercal}\mathbf{y}}\frac{\partial}{\partial y_k}u'\left(W\left(\mathbf{y}\right)\right)\right]$ and $\mathbb{E}_{\mathbf{x}}\left[\frac{\partial}{\partial x_k}u'\left(W\left(\mathbf{x}\right)\right)\right]$ (setting, for the latter term, $\theta=\mathbf{0}$ in Lemma A.4). The last equation gives

$$
\begin{aligned}
& \frac{\sqrt{T-t}P_0^2}{e^{\mu_n T+\sigma_n^{\intercal}\left(\beta-\frac{(T-t)\sigma_n}{2}\right)}}\frac{\partial p_n}{\partial \beta_k} \\
= \; & \sqrt{T-t}\sigma_{nk}\mathbb{E}_{\mathbf{y}}\left[u'\left(W\left(\mathbf{y}+\sqrt{T-t}\sigma_n\right)\right)\right]\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)\right] \\
& +\mathbb{E}_{\mathbf{y}}\left[y_k u'\left(W\left(\mathbf{y}+\sqrt{T-t}\sigma_n\right)\right)\right]\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)\right] \\
& -\mathbb{E}_{\mathbf{x}}\left[x_k u'\left(W\left(\mathbf{x}\right)\right)\right]\mathbb{E}_{\mathbf{y}}\left[u'\left(W\left(\mathbf{y}+\sqrt{T-t}\sigma_n\right)\right)\right] \\
= \; & \mathbb{E}_{\mathbf{y}}\left[\left(y_k+\sqrt{T-t}\sigma_{nk}\right)u'\left(W\left(\mathbf{y}+\sqrt{T-t}\sigma_n\right)\right)\right]\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)\right] \\
& -\mathbb{E}_{\mathbf{x}}\left[x_k u'\left(W\left(\mathbf{x}\right)\right)\right]\mathbb{E}_{\mathbf{y}}\left[u'\left(W\left(\mathbf{y}+\sqrt{T-t}\sigma_n\right)\right)\right] \qquad (27) \\
= \; & \mathbb{E}_{\widetilde{\mathbf{y}}}\left[\widetilde{y}_k u'\left(W\left(\widetilde{\mathbf{y}}\right)\right)\right]\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathbf{x}\right)\right)\right]-\mathbb{E}_{\mathbf{x}}\left[x_k u'\left(W\left(\mathbf{x}\right)\right)\right]\mathbb{E}_{\widetilde{\mathbf{y}}}\left[u'\left(W\left(\widetilde{\mathbf{y}}\right)\right)\right]
\end{aligned}
$$

where $\widetilde{\mathbf{y}}\sim\mathcal{N}\left(\sqrt{T-t}\sigma_n,\mathrm{I}_K\right)$ is independent of $\mathbf{x}$. That is,

$$
\begin{aligned}
& \frac{P_0^2\sqrt{T-t}\left(2\pi\right)^K}{e^{\mu_n T+\sigma_n^{\intercal}\left(\beta+\frac{(T-t)\sigma_n}{2}\right)}}\frac{\partial p_n}{\partial \beta_k} \\
= \; & \int_{\mathbb{R}^{2K}}u'\left(W\left(\widetilde{\mathbf{y}}\right)\right)u'\left(W\left(\mathbf{x}\right)\right)\widetilde{y}_k e^{-\frac{\left(\widetilde{\mathbf{y}}-\sqrt{T-t}\sigma_n\right)^{\intercal}\left(\widetilde{\mathbf{y}}-\sqrt{T-t}\sigma_n\right)+\mathbf{x}^{\intercal}\mathbf{x}}{2}}\mathrm{d}\mathbf{x}\mathrm{d}\widetilde{\mathbf{y}} \\
& -\int_{\mathbb{R}^{2K}}u'\left(W\left(\widetilde{\mathbf{y}}\right)\right)u'\left(W\left(\mathbf{x}\right)\right)x_k e^{-\frac{\left(\widetilde{\mathbf{y}}-\sqrt{T-t}\sigma_n\right)^{\intercal}\left(\widetilde{\mathbf{y}}-\sqrt{T-t}\sigma_n\right)+\mathbf{x}^{\intercal}\mathbf{x}}{2}}\mathrm{d}\mathbf{x}\mathrm{d}\widetilde{\mathbf{y}}
\end{aligned}
$$

Equivalently, changing variables of integration,

$$
\begin{aligned}
& \frac{P_0^2\sqrt{T-t}\left(2\pi\right)^K}{e^{\mu_n T+\sigma_n^{\intercal}\beta}}\frac{\partial p_n}{\partial \beta_k} \\
= \; & \int_{\mathbb{R}^{2K}}u'\left(W\left(\mathbf{y}\right)\right)u'\left(W\left(\mathbf{x}\right)\right)\left(y_k-x_k\right)e^{\sqrt{T-t}\sigma_n^{\intercal}\mathbf{y}-\frac{\mathbf{y}^{\intercal}\mathbf{y}+\mathbf{x}^{\intercal}\mathbf{x}}{2}}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \qquad (28)
\end{aligned}
$$

i.e,

$$
\frac{\partial p_n}{\partial \beta_k}=\frac{e^{\mu_n T+\sigma_n^{\intercal}\beta}}{P_0^2\sqrt{(T-t)}}\mathbb{E}_{(\mathbf{x},\mathbf{y})}\left[u'\left(W\left(\mathbf{x}\right)\right)u'\left(W\left(\mathbf{y}\right)\right)\left(y_k-x_k\right)e^{\sqrt{T-t}\sigma_n^{\intercal}\mathbf{y}}\right] \qquad (29)
$$

so that $\frac{\partial p_n}{\partial \beta_k}$ is directly proportional to the $2K$-dimensional integral in (28), which cannot be calculated analytically for general specifications of the functions $u(\cdot)$ and $\rho(\cdot)$. Yet, its integrand is symmetric with respect to the variables of integration in a way that allows the use of Lemma A.5. There are two cases to consider.

If $\sigma_n = \mathbf{0}$, the integral reads $\int_{\mathbb{R}^{2K}} g(\mathbf{x},\mathbf{y}) W(\mathbf{x},\mathbf{y}) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$ with $g : \mathbb{R}^{2K} \mapsto \mathbb{R}_{++}$ and $f : \mathbb{R}^{2K} \mapsto \mathbb{R}$ defined by

$$g(\mathbf{x},\mathbf{y}) = u'(W(\mathbf{x}))\, u'(W(\mathbf{y}))\, e^{-\frac{\mathbf{y}^\intercal \mathbf{y} + \mathbf{x}^\intercal \mathbf{x}}{2}} \qquad f(\mathbf{x},\mathbf{y}) = \mathbf{e}_k^\intercal (\mathbf{y} - \mathbf{x})$$

And, since $g(\mathbf{x},\mathbf{y})[W(\mathbf{x},\mathbf{y}) + W(\mathbf{y},\mathbf{x})] = 0\ \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^K$ while $g$ is symmetric, by Lemma A.5, the integral must be zero.

For $\sigma_n \neq \mathbf{0}$, observe that the quantity multiplying $\frac{\partial p_n}{\partial \beta_k}$ on the left-hand side of (28) is invariant with respect to $k \in \{1, ..., K\}$. Summing, therefore, gives

$$\begin{aligned}
& \frac{\sqrt{T-t}\,(2\pi)^K P_0^2}{e^{\mu_n T + \sigma_n^\intercal \left(\beta + \frac{(T-t)\sigma}{2}\right)}} \sum_{k=1}^{K} \sigma_{nk} \frac{\partial p_n}{\partial \beta_k} \\
= & \int_{\mathbb{R}^{2K}} u'(W(\mathbf{y}))\, u'(W(\mathbf{x}))\, e^{\sqrt{T-t}\sigma_n \mathbf{y} - \frac{\mathbf{y}^\intercal \mathbf{y} + \mathbf{x}^\intercal \mathbf{x}}{2}} \sum_{k=1}^{K} \sigma_{nk} (y_k - x_k)\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
= & \int_{\mathbb{R}^{2K}} g(\mathbf{x},\mathbf{y})\, h(\mathbf{x},\mathbf{y})\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}
\end{aligned}$$

with $g$ as before and $h : \mathbb{R}^{2K} \mapsto \mathbb{R}$ given by $h(\mathbf{x},\mathbf{y}) = \sigma_n^\intercal (\mathbf{y} - \mathbf{x})\, e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{y}}$. Lemma A.5 requires now that this integral is strictly positive since

$$\begin{aligned}
h(\mathbf{x},\mathbf{y}) + h(\mathbf{y},\mathbf{x}) &= \sigma_n^\intercal (\mathbf{y} - \mathbf{x}) \left( e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{y}} - e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{x}} \right) \\
&= e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{x}} \sigma_n^\intercal (\mathbf{y} - \mathbf{x}) \left( e^{\sqrt{T-t}\sigma_n^\intercal (\mathbf{y} - \mathbf{x})} - 1 \right) \geq 0 \quad \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^K
\end{aligned}$$

with the inequality strict on all of $\mathbb{R}^{2K}$ except for the zero-measure subset which consists of the vectors $(\mathbf{x},\mathbf{y})$: $\sigma_n^\intercal (\mathbf{y} - \mathbf{x}) = 0$.

**Supplementary Note for Theorem 1**

I will demonstrate briefly how Lemma A.1 can be applied in the opening section of the preceding proof. For any $(\beta, \mathbf{x}) \in \mathbb{R}^{2K}$, we have

$$\begin{aligned}
&\frac{\partial}{\partial \beta_k}\left(u'(W(\mathbf{x}))\, e^{\mu_n T + \sigma_n^\intercal\left(\beta + \sqrt{T-t}\mathbf{x}\right)}\right) \\
= \;& e^{\mu_n T + \sigma_n^\intercal\left(\beta + \sqrt{T-t}\mathbf{x}\right)}\left(\sigma_{nk} u'(W(\mathbf{x})) + u''(W(\mathbf{x}))\frac{\partial W(\mathbf{x})}{\partial \beta_k}\right) \\
&\frac{\partial^2}{\partial \beta_k^2}\left(u'(W(\mathbf{x}))\, e^{\mu_n T + \sigma_n^\intercal\left(\beta + \sqrt{T-t}\mathbf{x}\right)}\right) \\
= \;& e^{\mu_n T + \sigma_n^\intercal\left(\beta + \sqrt{T-t}\mathbf{x}\right)}\left(\begin{array}{c} \sigma_{nk}^2 u'(W(\mathbf{x})) \\ + u''(W(\mathbf{x}))\left(2\sigma_{nk}\frac{\partial W(\mathbf{x})}{\partial \beta_k} + \frac{\partial^2 W(\mathbf{x})}{\partial \beta_k^2}\right) \\ + u'''(W(\mathbf{x}))\left(\frac{\partial W(\mathbf{x})}{\partial \beta_k}\right)^2 \end{array}\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial \beta_k} u'(W(\mathbf{x})) &= u''(W(\mathbf{x}))\frac{\partial W(\mathbf{x})}{\partial \beta_k} \\
\frac{\partial^2}{\partial \beta_k^2} u'(W(\mathbf{x})) &= u''(W(\mathbf{x}))\frac{\partial^2 W(\mathbf{x})}{\partial \beta_k^2} + u'''(W(\mathbf{x}))\left(\frac{\partial W(\mathbf{x})}{\partial \beta_k}\right)^2
\end{aligned}$$

while, by (26),

$$\frac{\partial^2 W(\mathbf{x})}{\partial \beta_k^2} = \frac{\partial^2 \rho\left(\beta + \sqrt{T-t}\mathbf{x}\right)}{\partial \beta_k^2} + \sum_{i=1}^{J} \sigma_{ik}^2 e^{\mu_i T + \sigma_i^\intercal\left(\beta + \sqrt{T-t}\mathbf{x}\right)}$$

Fixing now $\beta_{-k} \in \mathbb{R}^{K-1}$, consider $W(\cdot)$ as a function of $\beta_k$ and $\mathbf{x}$:

$$W(\beta_k, \mathbf{x}) = \rho\left((\beta_k, \beta_{-k}) + \sqrt{T-t}\mathbf{x}\right) + \sum_{i=1}^{J} e^{\mu_i T + \sigma_i^\intercal\left((\beta_k, \beta_{-k}) + \sqrt{T-t}\mathbf{x}\right)}$$

Define also the function $H : \mathbb{R}^{K+1} \mapsto \mathbb{R}_{++}$ by

$$H(\beta_k, \mathbf{x}) = u'(W(\beta_k, \mathbf{x}))\, e^{\mu_n T + \sigma_n^\intercal\left((\beta_k, \beta_{-k}) + \sqrt{T-t}\mathbf{x}\right)}$$

For the utility functions $u(\cdot)$ that are generally of interest in financial economics, $H(\cdot)$ does satisfy the requirements of the lemma. □

For the remaining of this section, keep in mind (28). The derivative of interest has the same sign as the quantity

$$\begin{aligned}\delta_{nk} &= e^{-\frac{(T-1)\sigma_n^\intercal \sigma_n}{2}} \int_{\mathbb{R}^{2K}} u'(W(\mathbf{y}))(y_k - x_k)\, u'(W(\mathbf{x}))\, e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{y}} \mathrm{d}\Phi(\mathbf{x},\mathbf{y}) \\ &= \int_{\mathbb{R}^{2K}} u'\left(W\left(\mathbf{y} + \sqrt{T-t}\sigma_n\right)\right)\left(y_k + \sqrt{T-t}\sigma_{nk} - x_k\right) u'(W(\mathbf{x}))\, \mathrm{d}\Phi(\mathbf{x},\mathbf{y})\end{aligned}$$

the second equality applying Lemma A.2. When $\sigma_{nk} = 0$, this reads

$$\delta_{nk}^0 = \int_{\mathbb{R}^{2K}} u'\left(W\left(\mathbf{y} + \sqrt{T-t}\sigma_n\right)\right)(y_k - x_k)\, u'(W(\mathbf{x}))\, \mathrm{d}\Phi(\mathbf{x},\mathbf{y})$$

And if, in addition, $\sigma_n = \sigma_{nm}\mathbf{e}_m$, it further simplifies to

$$\delta_{nk}^* = \int_{\mathbb{R}^{2K}} u'\left(W\left(\mathbf{y} + \sqrt{T-t}\sigma_{nm}\mathbf{e}_m\right)\right)(y_k - x_k)\, u'(W(\mathbf{x}))\, \mathrm{d}\Phi(\mathbf{x},\mathbf{y})$$

**Proposition 3**

Observe first that the terminal wealth specification in (22) can be expressed as $W(\mathbf{x}) = W_1(\mathbf{x}_{-k}) + W_2(x_k)$ for some continuous functions $W_1 : \mathbb{R}^{K-1} \mapsto \mathbb{R}_{++}$ and $W_2 : \mathbb{R} \mapsto \mathbb{R}_{++}$. $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^K$ then, we have

$$\begin{aligned} u'(W(\mathbf{x})) &= \alpha\gamma e^{\alpha[W_1(\mathbf{x}_{-k}) + W_2(x_k)]} \\ u'\left(W\left(\mathbf{y} + \sqrt{T-t}\sigma_{nm}\mathbf{e}_m\right)\right) &= \alpha\gamma e^{\alpha\left[W_1\left(\mathbf{y}_{-k} + \sqrt{T-t}\sigma_{nm}\mathbf{e}_m\right) + W_2(y_k)\right]}\end{aligned}$$

(where now $\mathbf{e}_m \in \mathbb{R}^{K-1}$). Therefore,

$$\frac{\delta_{nk}^*}{\alpha^2\gamma^2} = \int_{\mathbb{R}^{2(K-1)}} \left(\int_{\mathbb{R}^2} (gf)(x_k, y_k)\, \mathrm{d}\Phi(x_k)\, \mathrm{d}\Phi(y_k)\right) h(\mathbf{x}_{-k}, \mathbf{y}_{-k})\, \mathrm{d}\Phi(\mathbf{x}_{-k})\, \mathrm{d}\Phi(\mathbf{y}_{-k})$$

where $f : \mathbb{R}^2 \mapsto \mathbb{R}$, $g : \mathbb{R}^2 \mapsto \mathbb{R}_{++}$, and $h : \mathbb{R}^{2(K-1)} \mapsto \mathbb{R}_{++}$ are given by

$$\begin{aligned} f(x_k, y_k) &= y_k - x_k \qquad g(x_k, y_k) = e^{\alpha[W_2(x_k) + F_k(y_k)]} \\ h(\mathbf{x}_{-k}, \mathbf{y}_{-k}) &= e^{\alpha\left[W_1\left(\mathbf{y}_{-k} + \sqrt{T-t}\sigma_{nm}\mathbf{e}_m\right) + W_1(\mathbf{x}_{-k})\right]}\end{aligned}$$

It is trivial now to verify that Lemma A.5 applies to the two-dimensional integral in the brackets, requiring it to be zero.

Turning to the terminal wealth specification in (19), observe that it can be written as $W(\mathbf{x}) = W_1(\mathbf{x}_{-m}) + W_2(x_m)$ for some continuous functions $W_1 : \mathbb{R}^{K-1} \mapsto \mathbb{R}_{++}$ and $W_2 : \mathbb{R} \mapsto \mathbb{R}_{++}$. $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^K$ again, we have

$$\begin{aligned} u'(W(\mathbf{x})) &= \alpha\gamma e^{\alpha[W_1(\mathbf{x}_{-m})+W_2(x_m)]} \\ u'\left(W\left(\mathbf{x}+\sqrt{T-t}\sigma_{nm}\mathbf{e}_m\right)\right) &= \alpha\gamma e^{\alpha\left[W_1(\mathbf{y}_{-m})+W_2\left(y_m+\sqrt{T-t}\sigma_{nm}\right)\right]} \end{aligned}$$

Hence,

$$\frac{\delta^*_{nk}}{\alpha^2\gamma^2} = \int_{\mathbb{R}^2} h(x_m, y_m)\left(\int_{\mathbb{R}^{2(K-1)}} (gf)(\mathbf{x}_{-m}, \mathbf{y}_{-m})\,\mathrm{d}\Phi(\mathbf{x}_{-m})\,\mathrm{d}\Phi(\mathbf{y}_{-m})\right)\mathrm{d}\Phi(x_m)\,\mathrm{d}\Phi(y_m)$$

with $f : \mathbb{R}^{2(K-1)} \mapsto \mathbb{R}_{++}$, $g : \mathbb{R}^{2(K-1)} \mapsto \mathbb{R}$, and $h : \mathbb{R}^2 \mapsto \mathbb{R}_{++}$ now given by

$$\begin{aligned} g(\mathbf{x}_{-m}, \mathbf{y}_{-m}) &= e^{\alpha[W_1(\mathbf{x}_{-m})+W_1(\mathbf{y}_{-m})]} \quad f(\mathbf{x}_{-m}, \mathbf{y}_{-m}) = (\mathbf{y}-\mathbf{x})\,\mathbf{e}_k = y_k - x_k \\ h(x_m, y_m) &= e^{\alpha\left[W_2(x_m)+W_2\left(y_m+\sqrt{T-t}\sigma_{nm}\right)\right]} \end{aligned}$$

Again by Lemma A.5, the $2(K-1)$-dimensional integral in the brackets must be zero. $\square$

To complete the analytical arguments that support the relevant discussion in the text, notice the following. Under the specification in (22), equation (6) reads

$$\begin{aligned} p_n &= e^{\mu_m T+\sigma_{nm}\left(\beta_m+\frac{(T-t)\sigma_{nm}}{2}\right)} \\ &\quad \frac{\mathbb{E}_{x_k}\left[e^{\alpha W_2(x_k)}\right]}{\mathbb{E}_{x_k}\left[e^{\alpha W_2(x_k)}\right]} \frac{\mathbb{E}_{\mathbf{x}_{-k}}\left[e^{\alpha W_1\left(\mathbf{x}_{-k}+\sqrt{T-t}\sigma_{nm}\mathbf{e}_m\right)}\right]}{\mathbb{E}_{\mathbf{x}_{-k}}\left[e^{\alpha W_1(\mathbf{x}_{-k})}\right]} \\ &= e^{\mu_m T+\sigma_{nm}\left(\beta_m+\frac{(T-t)\sigma_{nm}}{2}\right)} \frac{\mathbb{E}_{\mathbf{x}_{-k}}\left[e^{\alpha W_1\left(\mathbf{x}_{-k}+\sqrt{T-t}\sigma_{nm}\mathbf{e}_m\right)}\right]}{\mathbb{E}_{\mathbf{x}_{-k}}\left[e^{\alpha W_1(\mathbf{x}_{-k})}\right]} \qquad (30) \end{aligned}$$

Under (19), on the other hand, it gives

$$
\begin{aligned}
p_n &= e^{\mu_m T+\sigma_{nm}\left(\beta_m+\frac{(T-t)\sigma_{nm}}{2}\right)} \frac{\mathbb{E}_{x_m}\left[e^{\alpha W_2\left(x_m+\sqrt{T-t}\sigma_{nm}\right)}\right]}{\mathbb{E}_{x_m}\left[e^{\alpha W_2(x_m)}\right]} \frac{\mathbb{E}_{\mathbf{x}_{-m}}\left[e^{\alpha W_1(\mathbf{x}_{-m})}\right]}{\mathbb{E}_{\mathbf{x}_{-m}}\left[e^{\alpha W_1(\mathbf{x}_{-m})}\right]} \\
&= e^{\mu_m T+\sigma_{nm}\left(\beta_m+\frac{(T-t)\sigma_{nm}}{2}\right)} \frac{\mathbb{E}_{x_m}\left[e^{\alpha W_2\left(x_m+\sqrt{T-t}\sigma_{nm}\right)}\right]}{\mathbb{E}_{x_m}\left[e^{\alpha W_2(x_m)}\right]} \qquad (31)
\end{aligned}
$$

Finally, the wealth specification in (23) is a special case of either of (19)-(22) and can be written as $W(\mathbf{x}) = \sum_{i=1}^{K} W_i(x_i)$ for some continuous functions $W_i : \mathbb{R} \mapsto \mathbb{R}_{++}$. Setting now $m = n$ in (31), gives

$$
p_n = e^{\mu_n T+\sigma_{nn}\left(\beta_n+\frac{(T-t)\sigma_{nn}}{2}\right)} \frac{\mathbb{E}_{x_n}\left[e^{\alpha W_n\left(x_n+\sqrt{T-t}\sigma_{nn}\right)}\right]}{\mathbb{E}_{x_n}\left[e^{\alpha W_n(x_n)}\right]} \qquad (32)
$$

**Equation (17)**

By renaming the variables of integration, we can re-write $\delta^*_{nk}$ as follows

$$
\begin{aligned}
\delta^*_{nk} &= \int_{\mathbb{R}^{2K}} \left[\begin{array}{c} u'\left(W\left(\mathbf{y}+\sqrt{T-t}\sigma_n\right)\right) u'\left(W(\mathbf{x})\right) y_k \\ -u'\left(W\left(\mathbf{x}+\sqrt{T-t}\sigma_n\right)\right) u'\left(W(\mathbf{y})\right) y_k \end{array}\right] \mathrm{d}\Phi(\mathbf{x},\mathbf{y}) \\
&= \mathbb{E}_{y_k}\left[\mathbb{E}_{(\mathbf{x},\mathbf{y}_{-k})}\left[\left(\begin{array}{c} u'\left(W\left(\mathbf{y}+\sqrt{T-t}\sigma_n\right)\right) u'\left(W(\mathbf{x})\right) \\ -u'\left(W\left(\mathbf{x}+\sqrt{T-t}\sigma_n\right)\right) u'\left(W(\mathbf{y})\right) \end{array}\right)\right] y_k\right]
\end{aligned}
$$

The claim follows since $\mathbb{E}_{y_k}[y_k] = 0$.

**Proposition 2 and Corollary 2.1**

Recall that we have defined the index sets $K_n = \{m \in \{1,\ldots,K\} : \sigma_{nm} \neq 0\}$ and $N_k = \{n' \in \{1,\ldots,N\} : \sigma_{n'k} \neq 0\}$. Notice also that $n \notin N_k$ (since $\sigma_{nk} = 0$) while $M = |K_n| < K$. Now, by permuting if necessary the elements of the index set $\{1,\ldots,K\}$, it is without any loss of generality to take the first $M$ of these indices as the set $K_n$ and the last index to depict the $k$th dimension, the one under study.

In what follows, $\mathbf{x}_M \in \mathbb{R}^M$ is a collection of realizations for the increments of the first $M$ Brownian motions, $\{\beta_m(T) - \beta_m(t)\}_{m \in K_n}$. Similarly, albeit with a slight abuse of notation, $\mathbf{x}_{-M} \in \mathbb{R}^{K-M}$ depicts a collection of realizations for the Brownian increments $\{\beta_{k'}(T) - \beta_{k'}(t)\}_{k' \in \{1,\ldots,K\} \setminus K_n}$ which are listed, under the new indexing, as $M+1, \ldots, K$. Finally, $\mathbf{x}_{-(M,k)} \in \mathbb{R}^{K-M-1}$ refers to a collection of realizations for the increments of the Brownian motions in the set $\{1, \ldots, K\} \setminus (K_n \cup \{k\})$.

*Step 1.* Observe that

$$\mathbb{E}_{(\mathbf{x},\mathbf{y})}\left[u'(W(\mathbf{y}))(y_k - x_k)\, u'(W(\mathbf{x}))\right] = 0 \tag{33}$$

which, by renaming the variables $\mathbf{y}_M \in \mathbb{R}^M$, can be written also as

$$\mathbb{E}_{\mathbf{y}_{-(M,k)}}\left[\mathbb{E}_{(\mathbf{z}_M, y_k)}\left[u'(W(\mathbf{y}_{-M}, \mathbf{z}_M))\, \mathbb{E}_{\mathbf{x}}\left[(y_k - x_k)\, u'(W(\mathbf{x}))\,|y_k\right]|\mathbf{y}_{-M}\right]\right] = 0$$

Hence, the quantity $\delta^0_{nk}$ that we defined previously in this section is now given by

$$\begin{aligned}
& \delta^0_{nk} \\
= \; & \mathbb{E}_{(\mathbf{x},\mathbf{y})}\left[u'\left(W\left(\mathbf{y} + \sqrt{T-t}\sigma_n\right)\right)(y_k - x_k)\, u'(W(\mathbf{x}))\right] \\
= \; & \mathbb{E}_{(\mathbf{x},\mathbf{y})}\left[u'\left(W\left(\mathbf{y} + \sqrt{T-t}\sigma_n\right)\right)(y_k - x_k)\, u'(W(\mathbf{x}))\right] \\
& -\mathbb{E}_{\mathbf{y}_{-(M,k)}}\left[\mathbb{E}_{(\mathbf{z}_M, y_k)}\left[u'(W(\mathbf{y}_{-M}, \mathbf{z}_M))\, \mathbb{E}_{\mathbf{x}}\left[(y_k - x_k)\, u'(W(\mathbf{x}))\,|y_k\right]|\mathbf{y}_{-(M,k)}\right]\right] \\
= \; & \mathbb{E}_{\mathbf{y}_{-(M,k)}}\left[\mathbb{E}_{y_k}\left[\begin{pmatrix} \mathbb{E}_{\mathbf{y}_M}\left[u'\left(W\left(\mathbf{y} + \sqrt{T-t}\sigma_n\right)\right)|\mathbf{y}_{-(M,k)}\right] \\ -\mathbb{E}_{\mathbf{z}_M}\left[u'(W(\mathbf{y}_{-M}, \mathbf{z}_M))\,|\mathbf{y}_{-(M,k)}\right] \end{pmatrix} \mathbb{E}_{\mathbf{x}}\left[(y_k - x_k)\, u'(W(\mathbf{x}))\,|y_k\right]\right]\right] \\
= \; & \mathbb{E}_{\mathbf{y}_{-(M,k)}}\left[\mathbb{E}_{y_k}\left[\begin{pmatrix} e^{-\frac{(T-t)\sigma_n^\intercal \sigma_n}{2}} \mathbb{E}_{\mathbf{y}_M}\left[u'(W(\mathbf{y}))\, e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{y}}|\mathbf{y}_{-(M,k)}\right] \\ -\mathbb{E}_{\mathbf{z}_M}\left[u'(W(\mathbf{y}_{-M}, \mathbf{z}_M))\,|\mathbf{y}_{-(M,k)}\right] \end{pmatrix} \atop \mathbb{E}_{\mathbf{x}}\left[(y_k - x_k)\, u'(W(\mathbf{x}))\,|y_k\right]\right]\right] \\
= \; & \mathbb{E}_{\mathbf{y}_{-(M,k)}}\left[\mathbb{E}_{y_k}\left[\begin{matrix} \left(\dfrac{e^{-\frac{(T-t)\sigma_n^\intercal \sigma_n}{2}} \mathbb{E}_{\mathbf{y}_M}\left[u'(W(\mathbf{y}))e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{y}}|\mathbf{y}_{-(M,k)}\right]}{\mathbb{E}_{\mathbf{z}_M}\left[u'(W(\mathbf{y}_{-M}, \mathbf{z}_M))|\mathbf{y}_{-(M,k)}\right]} - 1\right) \\ \mathbb{E}_{\mathbf{z}_M}\left[u'(W(\mathbf{y}_{-M}, \mathbf{z}_M))\,|\mathbf{y}_{-(M,k)}\right] \mathbb{E}_{\mathbf{x}}\left[(y_k - x_k)\, u'(W(\mathbf{x}))\,|y_k\right] \end{matrix}\right]\right]
\end{aligned}$$

where the forth equality follows from Lemma A.2 while the last one from the fact that $\mathbf{y}_M$ lists exhaustively the Brownian dimensions that affect the $n$th terminal dividend.

*Step 2.* Fix now an arbitrary point $\mathbf{y}_{-(M,k)} \in \mathbb{R}^{K-M-1}$. I will show that the function $g_1 : \mathbb{R} \mapsto \mathbb{R}$ given by

$$g_1(y_k) = \frac{e^{-\frac{(T-t)\sigma_n^{\intercal}\sigma_n}{2}} \mathbb{E}_{\mathbf{y}_M}\left[u'(W(\mathbf{y}))\, e^{\sqrt{T-t}\sigma_n^{\intercal}\mathbf{y}}\right]}{\mathbb{E}_{\mathbf{z}_M}\left[u'(W(\mathbf{y}_{-M}, \mathbf{z}_M))\right]} - 1$$

is monotone under the conditions of the proposition. To this end, fix an arbitrary $y_k \in \mathbb{R}$. Since $\sigma_{nk} = 0$, $g_1'(y_k)$ has the same sign as the quantity

$$\begin{aligned} I(y_k) &= e^{-\frac{(T-t)\sigma_n^{\intercal}\sigma_n}{2}} \mathbb{E}_{(\mathbf{y}_M,\mathbf{z}_M)}\left[\begin{pmatrix} u''(W(\mathbf{y}))\, u'(W(\mathbf{y}_{-M}, \mathbf{z}_M))\, \frac{\partial W(\mathbf{y})}{\partial y_k} \\ -u'(W(\mathbf{y}))\, u''(W(\mathbf{y}_{-M}, \mathbf{z}_M))\, \frac{\partial W(\mathbf{y}_{-M},\mathbf{z}_M)}{\partial y_k} \end{pmatrix} e^{\sqrt{T-t}\sigma_n^{\intercal}\mathbf{y}}\right] \\ &= r_A e^{-\frac{(T-t)\sigma_n^{\intercal}\sigma_n}{2}} \\ &\quad \mathbb{E}_{(\mathbf{y}_M,\mathbf{z}_M)}\left[u'(W(\mathbf{y}))\, u'(W(\mathbf{y}_{-M}, \mathbf{z}_M)) \left(\frac{\partial W(\mathbf{y}_{-M}, \mathbf{z}_M)}{\partial y_k} - \frac{\partial W(\mathbf{y})}{\partial y_k}\right) e^{\sqrt{T-t}\sigma_n^{\intercal}\mathbf{y}}\right] \end{aligned}$$

Under the given terminal-wealth specification, though, we have

$$\begin{aligned} W(\mathbf{y}) &= \rho(\mathbf{y}_{-k}) + \sum_{n'=1}^{N} D_{n'}(\mathbf{y}) \\ &= \rho(\mathbf{y}_{-k}) + \sum_{n'=1}^{N} e^{\mu_{n'}T + \sigma_{n'}^{\intercal}\beta + \sqrt{T-t}\left(\sum_{k'\notin K_n} \sigma_{n'k'} y_{k'} + \sum_{m\in K_n} \sigma_{n'm} y_m\right)} \\ \frac{\partial W(\mathbf{y})}{\partial y_k} &= \sqrt{T-t} \sum_{n'\in N_k} \sigma_{n'k} e^{\mu_{n'}T + \sigma_{n'}^{\intercal}\beta + \sqrt{T-t}\left(\sum_{k'\notin K_n} \sigma_{n'k'} y_{k'} + \sum_{m\in K_n} \sigma_{n'm} y_m\right)} \end{aligned}$$

Hence,

$$\begin{aligned} &I(y_k) = r_A \sqrt{T-t}\, e^{-\frac{(T-t)\sigma_n^{\intercal}\sigma_n}{2}} \\ &\sum_{n'\in N_k} \sigma_{n'k} e^{\mu_{n'}T + \sigma_{n'}^{\intercal}\beta + \sqrt{T-t}\sum_{k'\notin K_n} \sigma_{n'k'} y_{k'}} \mathbb{E}_{(\mathbf{y}_M,\mathbf{z}_M)}\left[u'(W(\mathbf{y}))\, u'(W(\mathbf{y}_{-M}, \mathbf{z}_M))\, h_{n'}(\mathbf{y}_M, \mathbf{z}_M)\right] \end{aligned}$$

with $h_{n'}: \mathbb{R}^{2M} \mapsto \mathbb{R}$ defined as

$$\begin{aligned} h_{n'}\left(\mathbf{y}_M, \mathbf{z}_M\right) &= \left(e^{\sqrt{T-t}\sum_{m\in K_n}\sigma_{n'm}z_m} - e^{\sqrt{T-t}\sum_{m\in K_n}\sigma_{n'm}y_m}\right) e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{y}} \\ &= e^{\sqrt{T-t}\left(\sum_{m'\in K_n\setminus\left(K_n\cap K_{n'}\right)}\sigma_{nm'}y_{m'}+\sum_{m\in K_n\cap K_{n'}}\sigma_{nm}y_m\right)} \\ &\quad \left(e^{\sqrt{T-t}\sum_{m\in K_n\cap K_{n'}}\sigma_{n'm}z_m} - e^{\sqrt{T-t}\sum_{m\in K_n\cap K_{n'}}\sigma_{n'm}y_m}\right) \end{aligned}$$

where the second equality deploys the fact that $\sigma_{n'm} = 0\ \forall m \notin K_{n'}$. But, under condition (ii), $\sigma_{n'm} = \lambda_{n'}\sigma_{nm}\ \forall m \in K_n \cap K_{n'}$. Therefore,

$$\begin{aligned} h_{n'}\left(\mathbf{y}_M, \mathbf{z}_M\right) &= e^{\sqrt{T-t}\left(\sum_{m'\in K_n\setminus\left(K_n\cap K_{n'}\right)}\sigma_{nm'}y_{m'}+\sum_{m\in K_n\cap K_{n'}}\sigma_{nm}y_m\right)} \\ &\quad \left(e^{\lambda_{n'}\sqrt{T-t}\sum_{m\in K_n\cap K_{n'}}\sigma_{nm}z_m} - e^{\lambda_{n'}\sqrt{T-t}\sum_{m\in K_n\cap K_{n'}}\sigma_{nm}y_m}\right) \end{aligned}$$

If $\lambda_{n'} > 0$ ($\lambda_{n'} < 0$), then $h_{n'}\left(\mathbf{y}_M, \mathbf{z}_M\right) + h_{n'}\left(\mathbf{z}_M, \mathbf{y}_M\right) \leq 0$ ($\geq 0$) on $\mathbb{R}^{2M}$ with equality only on the zero-measure subset consisting of the vectors $\left(\mathbf{y}_M, \mathbf{z}_M\right) : \sum_{m\in K_n\cap K_{n'}}\sigma_{nm}\left(y_m - z_m\right) = 0$. By Lemma A.5, therefore, the typical expectation in the preceding sum is negative (positive) if $\lambda_{n'} > 0$ ($\lambda_{n'} < 0$).[39] Equivalently, the typical term of the sum is negative (positive) if $\lambda_{n'}\sigma_{n'k} > 0$ ($\lambda_{n'}\sigma_{n'k} < 0$). To sign the entire sum, it suffices that all of its terms are of the same sign. And this is guaranteed by condition (iii). To see this, consider the collection $\cup_{m\in K_n}N_m$ of those risky securities whose terminal dividend varies with at least one of the Brownian components that affect the $n$th dividend. Condition (ii) required a proportionality constant $\lambda_{n'}$ for those securities that are simultaneously members of this collection and of $N_k$: $n' \in \cup_{m\in K_n}\left(N_m \cap N_k\right)$. Clearly, if $\lambda_{n'}\sigma_{n'k}$ maintains the same sign on this set, $I\left(y_k\right)$ will have the opposite sign.

*Step 3.* Define the function $g_2 : \mathbb{R} \mapsto \mathbb{R}$ by

$$g_2\left(y_k\right) = \mathbb{E}_{\mathbf{z}_M}\left[u'\left(W\left(\mathbf{y}_{-M}, \mathbf{z}_M\right)\right)\right]\mathbb{E}_{\mathbf{x}}\left[\left(y_k - x_k\right)u'\left(W\left(\mathbf{x}\right)\right)\right]$$

[39]To use the lemma here, let $g := h_{n'}$ and define $f : \mathbb{R}^{2M} \mapsto \mathbb{R}_{++}$ by $f\left(\mathbf{y}_M, \mathbf{z}_M\right) = u'\left(W\left(\mathbf{y}\right)\right)u'\left(W\left(\mathbf{y}_{-M}, \mathbf{z}_M\right)\right)e^{-\frac{\mathbf{y}_M^\intercal\mathbf{y}_M+\mathbf{z}_M^\intercal\mathbf{z}_M}{2}}$.

Since $u'(\cdot) > 0$, Lemma A.6 ensures the existence of some $y_k^0 \in \mathbb{R}$ with $\left(y_k - y_k^0\right) g(y_k) > 0 \ \forall y_k \in \mathbb{R} \setminus \left\{y_k^0\right\}$.

*Step 4.* Let $\lambda_{n'}\sigma_{n'k} > 0$. By Step 2, $\lambda_{n'}\sigma_{n'k}g_1(\cdot)$ is strictly decreasing on $\mathbb{R}$. But then,

$$\begin{aligned} & \mathbb{E}_{y_k}\left[\lambda_{n'}\sigma_{n'k}g_1(y_k)\,g_2(y_k)\right] \\ < & \int_{y_k\in\left(y_k^0,+\infty\right)} \lambda_{n'}\sigma_{n'k}g_1\left(y_k^0\right) g_2(y_k)\,\mathrm{d}\Phi(y_k) + \int_{y_k\in\left(-\infty,y_k^0\right)} \lambda_{n'}\sigma_{n'k}g_1\left(y_k^0\right) g_2(y_k)\,\mathrm{d}\Phi(y_k) \\ = & \lambda_{n'}\sigma_{n'k}g_1\left(y_k^0\right)\mathbb{E}_{y_k}\left[g_2(y_k)\right] \end{aligned}$$

and, thus,

$$\begin{aligned} \lambda_{n'}\sigma_{n'k}\delta_{nk}^0 &= \mathbb{E}_{\mathbf{y}_{-(M,k)}}\left[\mathbb{E}_{y_k}\left[\lambda_{n'}\sigma_{n'k}g_1(y_k)\,g(y_k)\right]\right] \\ &< \lambda_{n'}\sigma_{n'k}g_1\left(y_k^0\right)\mathbb{E}_{\mathbf{y}_{-(M,k)}}\left[\mathbb{E}_{y_k}\left[g(y_k)\right]\right] = 0 \end{aligned}$$

the last equality following from (33). $\square$

Corollary 2.1 considers the case $K_n = \{m\}$. Condition (ii) of the proposition is now redundant while $K_n \backslash (K_n \cap K_{n'}) = \varnothing$. In Step 2, for any $n' \in N_k$, the relevant function reads $h_{n'}(y_m, z_m) = e^{\sqrt{T-t}\sigma_{nm}y_m}\left(e^{\sqrt{T-t}\sigma_{n'm}z_m} - e^{\sqrt{T-t}\sigma_{n'm}y_m}\right)$. This is zero if $n' \notin N_m$. For $n' \in N_k \cap N_m$, if $\sigma_{nm}\sigma_{n'm} > 0$ ($\sigma_{nm}\sigma_{n'm} < 0$), $h_{n'}(y_m, z_m) + h_{n'}(z_m, y_m)$ is non-positive (non-negative) on $\mathbb{R}^2$, being zero iff $y_m = z_m$. By Lemma A.5, therefore, the typical term in sum of $I(y_k)$ has the opposite (same) sign of (as) $\sigma_{n'k}$ if $\sigma_{nm}\sigma_{n'm} > 0$ ($\sigma_{nm}\sigma_{n'm} < 0$). Clearly, as long as $\sigma_{n'k}\sigma_{n'm}$ has the same sign across all $n' \in N_k \cap N_m$, $I(y_k)$ will have the opposite (same) sign of (as) $\sigma_{nm}$ if $\sigma_{n'k}\sigma_{n'm} > 0$ ($\sigma_{n'k}\sigma_{n'm} < 0$). Corollary 2.2 follows immediately, $N_k$ being a singleton.

**Proposition 1 and Corollary 1.1**

This proof proceeds in the same fashion as the preceding one.

*Step 1.* Fixing an arbitrary $\mathbf{y}_{-(M,k)} \in \mathbb{R}^{K-M-1}$, the function $g_1 : \mathbb{R} \mapsto \mathbb{R}$ is again strictly monotone, with $\sigma_{n'k}g'(y_k) > 0 \ \forall y_k \in \mathbb{R}$ in this case. To see this, observe that now

$$I(y_k) = e^{-\frac{(T-t)\sigma_n^\intercal \sigma_n}{2}} \mathbb{E}_{(\mathbf{y}_M, \mathbf{z}_M)} \left[ \begin{array}{c} u'(W(\mathbf{y}))\, u'(W(\mathbf{y}_{-M}, \mathbf{z}_M))\, e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{y}} \\ \left[ r_A(W(\mathbf{y}_{-M}, \mathbf{z}_M)) \frac{\partial W(\mathbf{y}_{-M}, \mathbf{z}_M)}{\partial y_k} - r_A(W(\mathbf{y})) \frac{\partial W(\mathbf{y})}{\partial y_k} \right] \end{array} \right]$$

Under the given terminal-wealth specification, the terminal-period endowment is a function $\rho(\mathbf{y}_{-k})$ while $\sigma_{n'm} = 0\ \forall (n', m) \in N_k \times K_n$. Condition (ii), moreover, requires that $\sigma_{n'm} = \sigma_{nm}$ for any $n'$ with $\sigma_{n'm} \neq 0$ for some $m \in K_n$. Hence,[40]

$$\begin{aligned} W(\mathbf{y}) = \rho(\mathbf{y}_{-k}) &+ \sum_{n'=1}^{N} D_{n'}(\mathbf{y}) \\ = \rho(\mathbf{y}_{-k}) &+ \sum_{n' \in \{1,\dots,N\} \setminus \cup_{m \in K_n} N_m} e^{\mu_{n'} T + \sigma_{n'}^\intercal \beta + \sqrt{T-t} \sum_{k' \notin K_n} \sigma_{n'k'} y_{k'}} \\ &+ e^{\sqrt{T-t} \sum_{m \in K_n} \sigma_{nm} y_m} \sum_{n' \in \cup_{m \in K_n} N_m} e^{\mu_{n'} T + \sigma_{n'}^\intercal \beta + \sqrt{T-t} \sum_{k' \notin K_n} \sigma_{n'k'} y_{k'}} \\ \frac{\partial W(\mathbf{y})}{\partial y_k} &= \sqrt{T-t} \sum_{n' \in N_k} \sigma_{n'k} e^{\mu_{n'} T + \sigma_{n'}^\intercal \beta + \sqrt{T-t} \sum_{k' \notin K_n} \sigma_{n'k'} y_{k'}} \\ &= \sqrt{T-t} \sum_{n' \in N_k} \sigma_{n'k} D_{n'}(\mathbf{y}_{-M}) \end{aligned}$$

Therefore,

$$\frac{e^{\frac{(T-t)\sigma_n^\intercal \sigma_n}{2}} I(y_k)}{\sqrt{T-t}} = \mathbb{E}_{(\mathbf{y}_M, \mathbf{z}_M)} \left[ u'(W(\mathbf{y}))\, u'(W(\mathbf{y}_{-M}, \mathbf{z}_M))\, g(\mathbf{y}_{-M}, \mathbf{z}_M) \right] \sum_{n' \in N_k} \sigma_{n'k} D_{n'}(\mathbf{y}_{-M})$$

[40] Some remarks about the way the terminal wealth is written out here. On the right-hand side of the second equality, I sum across the $N$ terminal dividends by partitioning them into two sets. The first summation collects the ones that are *not* correlated with any of the Brownian dimensions that affect the $n$th dividend. In the exponent of the typical term here, no terms of the form $\sigma_{n'm} y_m$ with $m \in K_n$ appear as they are all zero. The second summation collects the remaining dividends. In the exponent of the typical term now, there *are* terms of the form $\sigma_{n'm} y_m$ with $m \in K_n$. Yet, in all of them, $\sigma_{n'm} = \sigma_{nm}$ due to condition (ii). The product of the corresponding exponentials can be, therefore, pulled out of the summation. In the exponent of the typical term of the second summation, there can also be terms of the form $\sigma_{n'k'} y_{k'}$ with $k' \notin K_n$. The corresponding exponentials stay inside the summation. Observe finally that, under the assumed terminal wealth specification, no dividend $n'$ whose exponent includes the term $\sigma_{n'k} y_k$ is to be found in the second summation.

where $g : \mathbb{R}^{K+M} \mapsto \mathbb{R}$ is given by

$$g\left(\mathbf{y}_{-M},\left(\mathbf{y}_M,\mathbf{z}_M\right)\right)=\left[r_A\left(W\left(\mathbf{y}_{-M},\mathbf{z}_M\right)\right)-r_A\left(W\left(\mathbf{y}\right)\right)\right]e^{\sqrt{T-t}\sum_{m\in K_n}\sigma_{nm}y_m}$$

But

$$\begin{aligned} & g\left(\mathbf{y}_{-M},\left(\mathbf{y}_M,\mathbf{z}_M\right)\right)+g\left(\mathbf{y}_{-M},\left(\mathbf{z}_M,\mathbf{y}_M\right)\right) \\ = \; & \left[r_A\left(W\left(\mathbf{y}_{-M},\mathbf{z}_M\right)\right)-r_A\left(W\left(\mathbf{y}\right)\right)\right]\left(e^{\sqrt{T-t}\sum_{m\in K_n}\sigma_{nm}y_m}-e^{\sqrt{T-t}\sum_{m\in K_n}\sigma_{nm}z_m}\right)\end{aligned}$$

is non-negative on $\mathbb{R}^{2M}$, being zero only on the zero-measure set consisting of the vectors $\left(\mathbf{y}_M,\mathbf{z}_M\right) : \sum_{m\in K_n}\sigma_{nm}\left(y_m - z_m\right) = 0$.[41] This implies that the expectation above is positive (Lemma A.5), allowing in turn condition (iii) to ensure that $g_1'\left(y_k\right)$ has the same sign as $\sigma_{n'k}$ for any $n' \in N_k$.

*Step 2.* By the same argument as in the last two steps of the proof of Proposition 2, one can establish that $\sigma_{n'k}g_1\left(\cdot\right)$ is strictly increasing on $\mathbb{R}$ only if $\sigma_{n'k}\delta^0_{n'k} > 0$. □

For Corollary 1.1, let $K_n = \{m\}$. The requirements $\sigma_{n'm} = 0\ \forall\left(n',m\right) \in N_k \times K_n$ and $\sigma_{n'm} = \sigma_{nm}\ \forall m \in K_n\ \forall n' \in \cup_{m\in K_n}N_m$ reduce now, respectively, to $N_k \cap N_m = \varnothing$ and $\sigma_{n'm} = \sigma_{nm}\ \forall n' \in N_m$. The result reads $\sigma_{n'k}\delta^*_{n'k} > 0$.

[41] It is at this point of the proof that condition (ii) is deployed. For it allows the term $\eta = e^{\sqrt{T-t}\sum_{m\in K_n}\sigma_{nm}y_m}$ to be factored out of the second summation when the expression for the terminal wealth is written out. The condition ensures, therefore, that $\frac{\partial W(\mathbf{y})}{\partial\eta} > 0$ which, under DARA, implies in turn that $\frac{\partial r_A(W(\mathbf{y}))}{\partial\eta} < 0$.

**Claim 3.1**

Let $u(c) = \gamma c^{\alpha}$ $(\gamma, \alpha < 0)$. By condition (ii), we have $u'\left(W\left(\mathbf{x} + \sqrt{T-t}\sigma_n\right)\right) = \lambda_n^{\alpha-1} u'(W(\mathbf{x}))$. Hence, (27) now reads

$$
\begin{aligned}
\frac{\sqrt{T-t}P_0^2}{\lambda_n^{\alpha-1} e^{\mu_n T + \sigma_n^{\intercal}\left(\beta - \frac{(T-t)\sigma_n}{2}\right)}} \frac{\partial p_n}{\partial \beta_k} &= \mathbb{E}_{\mathbf{y}}\left[\left(y_k + \sqrt{T-t}\sigma_{nk}\right) u'(W(\mathbf{y}))\right] \mathbb{E}_{\mathbf{x}}\left[u'(W(\mathbf{x}))\right] \\
&\quad - \mathbb{E}_{\mathbf{x}}\left[x_k u'(W(\mathbf{x}))\right] \mathbb{E}_{\mathbf{y}}\left[u'(W(\mathbf{y}))\right] \\
&= \sqrt{T-t}\sigma_{nk} \mathbb{E}_{\mathbf{y}}\left[u'(W(\mathbf{y}))\right]^2 \\
&\quad + \mathbb{E}_{(\mathbf{x},\mathbf{y})}\left[u'(W(\mathbf{x})) u'(W(\mathbf{y}))(y_k - x_k)\right] \\
&= \sqrt{T-t}\sigma_{nk} \mathbb{E}_{\mathbf{y}}\left[u'(W(\mathbf{y}))\right]^2 = \sqrt{T-t}\sigma_{nk} P_0^2
\end{aligned}
$$

With $\alpha = 0$, this applies also when the utility function is logarithmic.

**Theorem 2**

Let $N = K$. As argued in the main text, in the economy I examine, dynamic completeness is equivalent to the $K \times K$ matrix $J_p(\omega, t)$ being nonsingular almost everywhere on $\Omega \times [0, T]$. In what follows, I establish that the latter condition obtains iff $\Sigma$ itself is nonsingular. In terms of notation, and no longer depicting the node $(\omega, t)$, $\mathbf{j}_{p,n}$ denotes the typical row of $J_p$ in vector form. Its typical entry $j_{p,(n,k)}$ is given by (28).

*Only If.* To establish the contrapositive statement, suppose that $\Sigma$ is singular. There exists, then, $\mathbf{v} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$ s.t. $\Sigma \mathbf{v} = \mathbf{0}$. Take now $\mathbf{a} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$ s.t. $\mathbf{a}^{\intercal}\mathbf{v} \neq 0$ and consider the hyperplane $H_{\mathbf{a}} = \left\{\mathbf{x} \in \mathbb{R}^K : \mathbf{a}^{\intercal}\mathbf{x} = 0\right\}$. For an arbitrary $\mathbf{x}_0 \in H_{\mathbf{a}}$, consider also the line through $\mathbf{x}_0$ in the direction of $\mathbf{v}$: $L(\mathbf{x}_0; \mathbf{v}) = \left\{\mathbf{x} \in \mathbb{R}^K : \mathbf{x} = \mathbf{x}_0 + t\mathbf{v}, t \in \mathbb{R}\right\}$. Since $\mathbf{v}$ and $H_{\mathbf{a}}$ are not parallel, $\mathbb{R}^K$ can be spanned as $\cup_{\mathbf{x}_0 \in H_{\mathbf{a}}} L(\mathbf{x}_0; \mathbf{v})$.[42] Hence, for the $n$th risky

[42] Let $\{\mathbf{v}_k\}_{k=1}^{K-1}$ be a basis for the hyperplane $H_{\mathbf{a}}$. As it is not collinear with $\mathbf{v}$, $\{\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_{K-1}\}$ is a basis of $\mathbb{R}^K$. Hence, any $\mathbf{x} \in \mathbb{R}^K$ can be written uniquely as $\mathbf{x} = \sum_{k=1}^{K-1} t_k \mathbf{v}_k + t\mathbf{v}$ for some $(t, t_1, \ldots, t_{K-1}) \in \mathbb{R}^K$. Equivalently, $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ for a unique $\mathbf{x}_0 = \sum_{k=1}^{K-1} t_k \mathbf{v}_k \in H_{\mathbf{a}}$.

security, we have

$$\begin{aligned}\frac{P_0^2\sqrt{T-t}\,(2\pi)^K}{e^{\mu_n T+\sigma_n^\intercal \beta}}\mathbf{a}^\intercal \mathbf{j}_{p,n} &= \int_{\mathbb{R}^{2K}} u'(F(\mathbf{x}))\,u'(F(\mathbf{y}))\,e^{\sigma_n^\intercal \mathbf{y}}\mathbf{a}^\intercal(\mathbf{y}-\mathbf{x})\,e^{-\frac{\mathbf{y}^\intercal\mathbf{y}+\mathbf{x}^\intercal\mathbf{x}}{2}}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\ &= \int_{H_{\mathbf{a}}^2} S(\mathbf{x}_0,\mathbf{y}_0;\mathbf{a})\,\mathrm{d}\mathbf{x}_0\mathrm{d}\mathbf{y}_0\end{aligned}$$

where $S : H_{\mathbf{a}}^2 \mapsto \mathbb{R}$ is given by

$$S(\mathbf{x}_0,\mathbf{y}_0;\mathbf{a}) = \int_{L^2(\mathbf{x}_0;\mathbf{v})} u'(F(\mathbf{x}))\,u'(F(\mathbf{y}))\,e^{\sigma_n^\intercal \mathbf{y}}\mathbf{a}^\intercal(\mathbf{y}-\mathbf{x})\,e^{-\frac{\mathbf{y}^\intercal\mathbf{y}+\mathbf{x}^\intercal\mathbf{x}}{2}}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$$

But $\sigma_n^\intercal \mathbf{x} = \sigma_n^\intercal \mathbf{x}_0\ \forall \mathbf{x} \in Ł(\mathbf{x}_0;\mathbf{v})$ and $\forall n = 1,\ldots,N$ so that the terminal-period wealth is a function of $\mathbf{x}_0$ rather than $\mathbf{x}$ on $L(\mathbf{x}_0;\mathbf{v})$. Moreover, $\mathbf{a}^\intercal\mathbf{x} = t\mathbf{a}^\intercal\mathbf{v}\ \forall \mathbf{x} \in Ł(\mathbf{x}_0;\mathbf{v})$. Hence, we may write

$$\begin{aligned}& S(\mathbf{x}_0,\mathbf{y}_0;\mathbf{a}) \\ =\ & \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0))\,u'(F(\mathbf{y}_0))\,e^{\sigma_n^\intercal \mathbf{y}_0}(t-\tau)\,\mathbf{a}^\intercal\mathbf{v}e^{-\frac{(\mathbf{y}_0+t\mathbf{v})^\intercal(\mathbf{y}_0+t\mathbf{v})+(\mathbf{x}_0+\tau\mathbf{v})^\intercal(\mathbf{x}_0+\tau\mathbf{v})}{2}}\mathrm{d}t\mathrm{d}\tau \\ =\ & u'(F(\mathbf{x}_0))\,u'(F(\mathbf{y}_0))\,e^{\sigma_n^\intercal \mathbf{y}_0}\int_{\mathbb{R}^2}\mathbf{a}^\intercal(t-\tau)\,\mathbf{v}e^{-\frac{(\mathbf{y}_0+t\mathbf{v})^\intercal(\mathbf{y}_0+t\mathbf{v})+(\mathbf{x}_0+\tau\mathbf{v})^\intercal(\mathbf{x}_0+\tau\mathbf{v})}{2}}\mathrm{d}t\mathrm{d}\tau \\ =\ & u'(F(\mathbf{x}_0))\,u'(F(\mathbf{y}_0))\,e^{\sigma_n^\intercal \mathbf{y}_0}\mathbb{E}_{(\mathbf{z},\widetilde{\mathbf{z}})}\left[\mathbf{a}^\intercal(\widetilde{\mathbf{z}}-\mathbf{z})\right] \\ =\ & u'(F(\mathbf{x}_0))\,u'(F(\mathbf{y}_0))\,e^{\sigma_n^\intercal \mathbf{y}_0}\mathbf{a}^\intercal(\mathbf{x}_0-\mathbf{y}_0) = 0\end{aligned}$$

where $(\mathbf{z},\widetilde{\mathbf{z}}) \sim \mathcal{N}\left(-(\mathbf{x}_0,\mathbf{y}_0),\begin{pmatrix}\mathbf{v}\mathbf{v}^\intercal & \mathbf{0}\\ \mathbf{0} & \mathbf{v}\mathbf{v}^\intercal\end{pmatrix}\right)$. And as the choices of stock and Brownian node were arbitrary in this argument, we have just established that $\mathbf{a}^\intercal\mathbf{j}_{p,n}(\omega,t) = 0$ for all $n = 1,\ldots,K$ and all $(\omega,t) \in \Omega\times[0,T]$. The Jacobian $J_p(\omega,t)$ is indeed singular everywhere on $\Omega\times[0,T]$.

*If.* For any $\mathbf{v} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$, the non-singularity of $\Sigma$ guarantees at least one nonzero entry for the vector $\Sigma\mathbf{v}$. Let it be the $n$th one: $\sigma_n^\intercal\mathbf{v} = \nu \neq 0$. Switching the vectors $\mathbf{a}$ and $\sigma_n$ in the geometric argument made for the preceding part, we have

$$\frac{P_0^2\sqrt{T-t}\,(2\pi)^K}{e^{\mu_n T+\sigma_n^\intercal \beta}}\mathbf{v}^\intercal\mathbf{j}_n = \int_{H_{\sigma_n}^2} S(\mathbf{x}_0,\mathbf{y}_0;\mathbf{v})\,\mathrm{d}\mathbf{x}_0\mathrm{d}\mathbf{y}_0$$

Yet, now $\mathbf{v}^\intercal\mathbf{x} = \mathbf{v}^\intercal\mathbf{x}_0 + t\mathbf{v}^\intercal\mathbf{v}$ and $\sigma_n^\intercal\mathbf{x} = t\nu$ $\forall\mathbf{x} \in L(\mathbf{x}_0;\mathbf{v})$ and $\forall n = 1,\ldots,K$ so that

$$\begin{aligned}
& S(\mathbf{x}_0,\mathbf{y}_0;\mathbf{v}) \\
= \; & \mathbf{v}^\intercal(\mathbf{y}_0 - \mathbf{x}_0) \\
& \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0 + \tau\mathbf{v}))\, u'(F(\mathbf{y}_0 + t\mathbf{v}))\, e^{t\nu} e^{-\frac{(\mathbf{y}_0+t\mathbf{v})^\intercal(\mathbf{y}_0+t\mathbf{v}) + (\mathbf{x}_0+\tau\mathbf{v})^\intercal(\mathbf{x}_0+\tau\mathbf{v})}{2}} \mathrm{d}t\mathrm{d}\tau \\
& +\mathbf{v}^\intercal\mathbf{v} \\
& \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0 + \tau\mathbf{v}))\, u'(F(\mathbf{y}_0 + t\mathbf{v}))\, e^{t\nu}(t-\tau)\, e^{-\frac{(\mathbf{y}_0+t\mathbf{v})^\intercal(\mathbf{y}_0+t\mathbf{v}) + (\mathbf{x}_0+\tau\mathbf{v})^\intercal(\mathbf{x}_0+\tau\mathbf{v})}{2}} \mathrm{d}t\mathrm{d}\tau \\
= \; & \mathbf{v}^\intercal(\mathbf{y}_0 - \mathbf{x}_0)\, \mathbb{E}_{(\mathbf{z},\widetilde{\mathbf{z}})}\left[e^{\frac{\nu(\widetilde{z}_n - y_{0n})}{v_n}} u'(F(\mathbf{z}))\, u'(F(\widetilde{\mathbf{z}}))\right] \\
& +\mathbf{v}^\intercal\mathbf{v} \\
& \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0 + \tau\mathbf{v}))\, u'(F(\mathbf{y}_0 + t\mathbf{v}))\, e^{t\nu}(t-\tau)\, e^{-\frac{(\mathbf{y}_0+t\mathbf{v})^\intercal(\mathbf{y}_0+t\mathbf{v}) + (\mathbf{x}_0+\tau\mathbf{v})^\intercal(\mathbf{x}_0+\tau\mathbf{v})}{2}} \mathrm{d}t\mathrm{d}\tau
\end{aligned}$$

where $(\mathbf{z},\widetilde{\mathbf{z}}) \sim \mathcal{N}\left(-(\mathbf{x}_0,\mathbf{y}_0), \begin{pmatrix} \mathbf{v}\mathbf{v}^\intercal & \mathbf{0} \\ \mathbf{0} & \mathbf{v}\mathbf{v}^\intercal \end{pmatrix}\right)$.

There are two cases to consider, depending on whether or not $\mathbf{v}$ and $\sigma_n$ are collinear. If they are, then $\mathbf{v}^\intercal\mathbf{x}_0 = \mathbf{v}^\intercal\mathbf{y}_0 = 0$. Otherwise, we can span $\mathbb{R}^K$ as $\cup_{r\in\mathbb{R}} \cup_{\mathbf{x}_0 \in H_{(\sigma_n;r)}} L(\mathbf{x}_0;\mathbf{v})$ where $H_{(\sigma_n;r)} = \{\mathbf{x}_0 \in H_{\sigma_n} : \mathbf{v}^\intercal\mathbf{x}_0 = r\}$. In this case, we may write $\mathbf{v}^\intercal\mathbf{j}_{p,n} = \int_{\mathbb{R}}\left(\int_{H^2_{(\sigma_n;r)}} S(\mathbf{x}_0,\mathbf{y}_0;\mathbf{v})\,\mathrm{d}\mathbf{x}_0\mathrm{d}\mathbf{y}_0\right)\mathrm{d}r$. Regarding, though, the integration in the brackets, in the expansion for $S(\mathbf{x}_0,\mathbf{y}_0;\mathbf{v})$ above we now have $\mathbf{v}^\intercal(\mathbf{y}_0 - \mathbf{x}_0) = r - r$.

In either case, therefore, $\mathbf{v}^\intercal(\mathbf{y}_0 - \mathbf{x}_0) = 0$ and, thus,

$$\begin{aligned}
& S(\mathbf{x}_0,\mathbf{y}_0;\mathbf{v}) \\
= \; & \mathbf{v}^\intercal\mathbf{v} \\
& \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0 + \tau\mathbf{v}))\, u'(F(\mathbf{y}_0 + t\mathbf{v}))\, e^{t\nu}(t-\tau)\, e^{-\frac{(\mathbf{y}_0+t\mathbf{v})^\intercal(\mathbf{y}_0+t\mathbf{v}) + (\mathbf{x}_0+\tau\mathbf{v})^\intercal(\mathbf{x}_0+\tau\mathbf{v})}{2}} \mathrm{d}t\mathrm{d}\tau \\
= \; & \mathbf{v}^\intercal\mathbf{v} \\
& \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0 + t\mathbf{v}))\, u'(F(\mathbf{y}_0 + \tau\mathbf{v}))\, e^{\tau\nu}(\tau - t)\, e^{-\frac{(\mathbf{y}_0+\tau\mathbf{v})^\intercal(\mathbf{y}_0+\tau\mathbf{v}) + (\mathbf{x}_0+t\mathbf{v})^\intercal(\mathbf{x}_0+t\mathbf{v})}{2}} \mathrm{d}t\mathrm{d}\tau
\end{aligned}$$

Writing now $S(\mathbf{x}_0, \mathbf{y}_0; \mathbf{v})$ and $S(\mathbf{y}_0, \mathbf{x}_0; \mathbf{v})$ by the first and second of these equalities, respectively, gives

$$S(\mathbf{x}_0, \mathbf{y}_0; \mathbf{v}) + S(\mathbf{y}_0, \mathbf{x}_0; \mathbf{v}) = \mathbf{v}^\intercal \mathbf{v} \int_{\mathbb{R}^2} g(\mathbf{x}_0 + \tau\mathbf{v}, \mathbf{y}_0 + t\mathbf{v}) \left(e^{t\nu} - e^{\tau\nu}\right)(t - \tau)\, \mathrm{d}t\mathrm{d}\tau$$

with $g : \mathbb{R}^{2K} \mapsto \mathbb{R}_{++}$ defined by

$$g(\mathbf{x}_0 + \tau\mathbf{v}, \mathbf{y}_0 + t\mathbf{v}) = u'(F(\mathbf{x}_0 + \tau\mathbf{v}))\, u'(F(\mathbf{y}_0 + t\mathbf{v}))\, e^{-\frac{(\mathbf{y}_0 + t\mathbf{v})^\intercal(\mathbf{y}_0 + t\mathbf{v}) + (\mathbf{x}_0 + \tau\mathbf{v})^\intercal(\mathbf{x}_0 + \tau\mathbf{v})}{2}}$$

Now, since $\nu \neq 0$, $\nu\left(e^{t\nu} - e^{\tau\nu}\right)(t - \tau) > 0$ $\forall t, \tau \in \mathbb{R}$ apart from the zero-measure subset $(t, \tau)$: $t = \tau$. Given also that $\mathbf{v}^\intercal\mathbf{v} > 0$, Lemma A.5 in Appendix A implies that $S(\mathbf{x}_0, \mathbf{y}_0; \mathbf{v}) + I(\mathbf{y}_0, \mathbf{x}_0; \mathbf{v})$ has the same sign as $\nu$. And so must do, of course, the quantity $2\mathbf{v}^\intercal\mathbf{j}_{p,n}$.
Which proves that $J_p(\omega, t)$ is non-singular everywhere on $\Omega \times [0, T]$. For we have established that, at an arbitrary $(\omega, t)$ and for an arbitrary $\mathbf{v} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$, the vector $J_p(\omega, t)\,\mathbf{v}$ has at least one nonzero entry.

# D Dividend-financed Intermediate Consumption

Let $f : \Omega \times [t, T] \mapsto \mathbb{R}$ be a stochastic process with $f(\omega, s) = f(\mathcal{I}(\omega, s))$. Let also $t = s_0 < s_1 < \cdots < s_{n-1} < s_m = T$ be a partition of $[t, T]$ and $\Delta_i = s_i - s_{i-1}$ for $i = 1, \ldots, m$. Given any $\omega \in \Omega$, as long as the time-paths $f(\omega, \cdot)$ are continuous, their time-integral can be approximated using Riemann-Stieltjes sums: $\int_t^T f(\omega, s)\, \mathrm{d}s = \lim_{m \to +\infty} \sum_{i=1}^m f(\omega, s_{i-1})\, \Delta_i$.[43] Fixing the arbitrary state, we may dismiss it from our notation henceforth. As the increments of the Brownian process are independent, for each $m$ in the

[43] It is well-known that a set of sufficient conditions for the integral $\int_t^T f(\omega, s)\, \mathrm{d}g(s)$ to exist in the Riemann-Stieltjes sense is for (i) $f(\omega, \cdot)$ and $g(\cdot)$ to not have discontinuities at the same point of $[t, T]$ and (ii) $f(\omega, \cdot)$ to be continuous and $g(\cdot)$ to have bounded variation. Here, $g(\cdot)$ being the identity function, it is everywhere continuous and has bounded variation (in fact, $\sum_{i=1}^m |g(s_i) - g(s_{i-1})| = T - t$ does not even depend on the interval partition). Clearly, (i)-(ii) are immediately satisfied if $f(\omega, \cdot)$ is continuous.

approximating sequence, we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{n} f\left(\mathcal{I}\left(s_{i-1}\right)\right)\Delta_i|\mathcal{F}_t\right] &= \sum_{i=1}^{n}\mathbb{E}\left[f\left(\mathcal{I}\left(s_{i-1}\right)\right)|\mathcal{F}_{s_0}\right]\Delta_i \\
= \quad & \sum_{i=1}^{n}\mathbb{E}\left[f\left(\beta\left(s_0\right)+\sum_{j=0}^{i-1}\beta\left(s_{j+1}\right)-\beta\left(s_j\right), s_{i-1}\right)|\mathcal{F}_{s_0}\right]\Delta_i \\
= \quad & \sum_{i=1}^{n}\mathbb{E}\left[f\left(\beta\left(s_0\right)+\sum_{j=0}^{i-1}\mathbf{x}_j, s_{i-1}\right)|\mathcal{F}_{s_0}\right]\Delta_i \\
= \quad & \sum_{i=1}^{n}\mathbb{E}\left[f\left(\beta\left(s_0\right)+\mathbf{y}_{i-1}, s_{i-1}\right)|\mathcal{F}_{s_0}\right]\Delta_i
\end{aligned}$$

with the $\mathbf{x}_j$'s independently distributed $\mathcal{N}\left(\mathbf{0},\Delta_{j+1}\mathrm{I}_K\right)$ and, consequently, $\mathbf{y}_{i-1}\sim\mathcal{N}\left(\mathbf{0},\left(s_i-s_0\right)\mathrm{I}_K\right)$ for their sum. In the limit, therefore, as $m\to+\infty$

$$(*)\ \mathbb{E}\left[\int_t^T f\left(\mathcal{I}\left(s\right)\right)\mathrm{d}s|\mathcal{F}_t\right]=\int_t^T\mathbb{E}\left[f\left(\beta\left(t\right)+\mathbf{y}\left(s\right),s\right)|\mathcal{F}_t\right]\mathrm{d}s$$

where $\mathbf{y}\left(s\right)\sim\mathcal{N}\left(\mathbf{0},\left(s-t\right)\mathrm{I}_K\right)$.

Suppose also that, for the arbitrary Brownian component $\beta_k\left(t\right)$, the derivative $\frac{\partial f(\mathcal{I}(s))}{\partial\beta_k(t)}$ exists and is continuous $\forall\in\left[t,T\right]$. As long as it commutes in the expectation operator, for each $m$ in the approximating sequence above, we get

$$\frac{\partial\mathbb{E}\left[\sum_{i=1}^{m} f\left(\mathcal{I}\left(s_{i-1}\right)\right)\Delta_i|\mathcal{F}_t\right]}{\partial\beta_k\left(t\right)}=\sum_{i=1}^{m}\frac{\partial\mathbb{E}\left[f\left(\beta\left(s_0\right)+\mathbf{y}_{i-1},s_{i-1}\right)|\mathcal{F}_{s_0}\right]}{\partial\beta_k\left(s_0\right)}\Delta_i$$

and, as $m\to+\infty$,

$$(**)\ \frac{\partial}{\partial\beta_k\left(t\right)}\int_t^T\mathbb{E}\left[f\left(\mathcal{I}\left(s\right)\right)|\mathcal{F}_t\right]\mathrm{d}s=\int_t^T\frac{\partial\mathbb{E}\left[f\left(\beta\left(t\right)+\mathbf{y}\left(s\right),s\right)|\mathcal{F}_t\right]}{\partial\beta_k\left(t\right)}\mathrm{d}s$$

For $n\in\{0,1,\ldots,N\}$, define now $f_n:\Omega\times[t,T]\mapsto\mathbb{R}$ by $f_0\left(s\right)=u'\left(W\left(\mathcal{I}\left(s\right)\right)\right)$ and, for $n\geq1$, $f_n\left(s\right)=u'\left(W\left(\mathcal{I}\left(s\right)\right)\right)D_n\left(W\left(\mathcal{I}\left(s\right)\right)\right)$. As long as $u\left(\cdot\right)$ and $D_n\left(\cdot\right)$ are, respectively, continuously-differentiable and continuous and

Lemma A.1 applies, $(*)$ and $(**)$ give, respectively,

$$P_n(t) = \int_t^T P_{n,s}(t)\,\mathrm{d}s \quad \text{and} \quad \frac{\partial P_n(t)}{\partial \beta_k(t)} = \int_t^T \frac{\partial P_{n,s}(t)}{\partial \beta_k(t)}\mathrm{d}s$$

where $P_{n,s}(t)$ is the absolute price I have analyzed in this paper taking $s$ to be the terminal date. But then, by (9), we ought to have

$$\begin{aligned}
P_0(t)^2 \frac{\partial p_n(t)}{\partial \beta_k(t)} &= P_0(t)\frac{\partial P_n(t)}{\partial \beta_k(t)} - P_n(t)\frac{\partial P_0(t)}{\partial \beta_k(t)} \\
&= \int_t^T \left( \begin{array}{c} \mathbb{E}\left[f\left(\beta(t) + \mathbf{y}(s), s\right)|\mathcal{F}_t\right] \frac{\partial \mathbb{E}[g(\beta(t)+\widetilde{\mathbf{y}}(s),s)|\mathcal{F}_t]}{\partial \beta_k(t)} \\ -\mathbb{E}\left[g\left(\beta(t) + \widetilde{\mathbf{y}}(s), s\right)|\mathcal{F}_t\right] \frac{\partial \mathbb{E}[f(\beta(t)+\mathbf{y}(s),s)|\mathcal{F}_t]}{\partial \beta_k(t)} \end{array} \right) \mathrm{d}s \\
&= \int_t^T \left(P_{0,s}(t)\frac{\partial P_{n,s}(t)}{\partial \beta_k(t)} - P_{n,s}(t)\frac{\partial P_{0,s}(t)}{\partial \beta_k(t)}\right)\mathrm{d}s \\
&= \int_t^T P_{0,s}(t)^2 \frac{\partial p_{n,s}(t)}{\partial \beta_k(t)}\mathrm{d}s
\end{aligned}$$

with $\widetilde{\mathbf{y}}(s) \sim \mathcal{N}(\mathbf{0}, (s-t)\,\mathrm{I}_K)$, independent of $\mathbf{y}(s)$.

To complete the argument, recall that each and every result in the paper obtains through signing the integrand term $\frac{\partial p_{n,s}(t)}{\partial \beta_k(t)}$ of the last integral above, taking $s$ as the terminal date. And as the matrix of factor loadings $\Sigma$ is constant, so is the respective sign on $[t, T]$. Being in fact the sign of the integral, all of my results remain valid when intermediate consumption is dividend-financed. Obviously, this is still the case as $T \to \infty$.